# The Lesser of Two Evils*

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**Abstract**

When a single indivisible good must be allocated among agents, no allocation is ideal and envy is unavoidable. Standard fairness criteria, including envy-freeness, are largely silent in such environments. This paper introduces Consensus Lesser Evil, a new criterion for ranking imperfect allocations based on the unanimity logic underlying the Pareto principle, extended to judgments of extended sympathy. The criterion is ordinal and pluralistic: the social ranking respects cross-agent comparisons only when both agents agree on them, requiring no exogenous interpersonal utility comparisons. Our main result is a representation theorem characterizing all leximin-compatible social preferences satisfying Consensus Lesser Evil in a two-agent economy. We show that any such preference admits a simple welfare representation: the welfare of each agent is given by the sum of utility indices measuring the desirability of their bundle as evaluated through each agent's preferences. We offer several interpretations of this measure. We then study the relationship between Consensus Lesser Evil, envy-freeness, and the grading principle. We establish that Consensus Lesser Evil and envy-freeness are logically independent, and characterize preferences satisfying both simultaneously. We further show that Consensus Lesser Evil refines the grading principle and is equivalent, on the Pareto frontier, to its symmetric, consensus-based extension. Finally, we demonstrate that the two criteria provide complementary normative guidance: Consensus Lesser Evil imposes strictly tighter restrictions precisely in the environments where envy-freeness is entirely silent, confirming its role as the criterion of last resort for the lesser evil problem.

Keywords: interpersonal comparisons; partial comparability; envy-freeness; social welfare; leximin; Rawlsian priority; public justification.

JEL codes: D63; D71; D60; D81; I31.

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# Introduction

Consider a transplant committee who has to allocate a single organ between two equally qualified patients. Since only one of them can be treated, no allocation is ideal and, in particular, envy is unavoidable. The normative question is therefore not which allocation is best, but which imperfect compromise constitutes the "lesser evil." Existing fairness ordinal criteria are largely silent on this question. Envy-freeness (Foley, 1967; Varian, 1974; Kolm, 1972) can only rank allocations that are envy-free above those that are not.

This paper introduces a new criterion, *Consensus Lesser Evil*, which retains normative bite in these situations. This criterion extends the unanimity logic underlying the Paretian criterion to what Arrow (1963) terms "extended sympathy" judgments. The Pareto principle derives its normative force from unanimity alone; requiring no interpersonal comparisons, it remains silent on distributive justice. Yet individuals do compare their situation to that of others, thereby providing a natural, albeit partial, informational basis for cross-agent evaluations (Arrow, 1977; Sen, 1970). Formally, consider two Pareto-efficient allocations $f$ and $g$ in a two-agent economy. *Consensus Lesser Evil*[1] requires that the social ranking respect cross-agent comparisons on which both agents agree: if agent $i$ and agent $j$ both rank $f_i$ above $g_j$, the social ranking should reflect that unanimous judgment. The criterion is ordinal and pluralistic: it compares no exogenous utility numbers across agents, and both agents' perspectives enter matter in the ranking. Through this criterion, we thus ask what welfare measures emerge when the unanimity logic is extended to judgments of extended sympathy.

Our main result characterizes the class of social preference representations compatible with *Consensus on the Lesser Evil* under the leximin. To our knowledge, no analogous representation exists for the most prominent criteria in the literature: envy-freeness or the grading principle (Suppes, 1966; Sen, 1970). The representation theorem reveals that a surprisingly simple and sound class of utility indices characterizes the criterion. Let $m_i^j(f_i)$ denote an index capturing how desirable agent $i$'s bundle under allocation $f$ appears when evaluated through the preferences of agent $j$. For every Pareto-efficient allocation $f = (f_i, f_{-i})$, impose the normalization $m_i^i(f_i) = m_{-i}^i(f_i)$, so that $m_j^i$ represents the utility index associated with agent $i$, taking $j$ as the benchmark. The key finding is that, for any leximin-representable social preference satisfying Consensus on the Lesser Evil, the social planner can be interpreted as aggregating benchmarks across agents. Specifically, the welfare of agent $i$ is given by the score $m_i^1(f_i) + m_i^2(f_i)$. We offer several interpretation for this measure of welfare.

This result is obtained in two-agent economy under the maintained assumption that social preferences are compatible with the leximin ordering. Leximin is the natural companion to this criterion: both are animated by the commitment to find a second best allocation when complete fairness is unfeasible (Rawls, 1971; Sen, 1970). For the sake of completness, we provide an axiomatization adapting Hammond (1976) to our setting at the end of the paper. The two-agent case is the natural building block for extensions to larger economies, paralleling the situation for both envy-freeness and the grading principle, each of which admits multiple non-equivalent generalizations beyond two agents (Varian, 1974; Thomson, 1982; Sen, 1970).[2] At the end of the paper, we discuss the results and outline possible extensions to larger economies.

Our main result also helps clarify the relationship between Consensus Lesser Evil, envy-

[1] The criterion is named to reflect both the method and the domain of application. *Consensus* captures the epistemological basis for the ranking: we rank one imperfect outcome above another only when both agents unanimously agree on the comparison, so that the ranking rests on no contested interpersonal weight. *Lesser Evil* is designed for, but not reduced to, environments where the first-best is unattainable and where existing criteria are silent. The idea that rational choice among imperfect alternatives reduces to identifying the lesser evil has been formalized in the context of non-ideal justice as the problem of ranking feasible but second-best outcomes (Rawls, 1971; Sen, 2009). Consensus Lesser Evil brings this logic into the social choice setting. The name refers to the structure of the problem—unavoidable imperfection—not to any particular political philosophy.

[2] While Varian (1974) requires for envy-freeness that no agent envy one another, Thomson (1982) argues that it makes more sense to demand that every agent be better off with what they have than with what other individuals have on average.

freeness, and the grading principle. We study this relationship along two lines. *First,* we characterize the logical relationships between these criteria. We show that envy-freeness and Consensus Lesser Evil are logically independent—neither implies the other—and characterize the class of social preferences satisfying both simultaneously. We then turn to the grading principle. Unlike Consensus Lesser Evil, the grading principle evaluates allocations from the standpoint of a single agent. We establish that the social preference represented by $(m_i^i, m_j^i)$ refines the grading principle applied from the perspective of agent $j$. Our analysis thereby provides the first representation result for a social preference based on a refinement of the grading principle. We further establish that, when restricted to Pareto-efficient allocations, Consensus Lesser Evil is equivalent to a condition that symmetrically aggregates the binary comparisons underlying the grading principle. In this sense, the criterion is precisely the symmetric, consensus-based extension of the grading principle that the unanimity motivation demands.

*Second,* we examine how the normative bite of envy-freeness and Consensus Lesser Evil varies with the structure of the feasible set. The lesser evil problem is most acute when the feasible set structurally favors one agent and envy cannot be eliminated. We show that Consensus Lesser Evil remains operative in exactly these environments, imposing strictly tighter restrictions on the social ranking than envy-freeness, which is entirely silent. Conversely, when the feasible set consists only of envy-free allocations, envy-freeness is the more restrictive criterion. This comparison maps out the domains in which each criterion provides stronger normative guidance, and confirms that they are genuinely complementary: Consensus Lesser Evil is the criterion of last resort for the lesser evil problem, providing normative traction precisely where envy-freeness cannot.

The concepts developed in the first two sections are illustrated using a common running example. Section 1 introduces the framework and pseudo-egalitarian social preferences, represented by the leximin criterion. Section 2 introduces the Consensus on the Lesser Evil, derives its characterizing representation, and discusses its interpretation. Section 3 examines the practical implications of our results. Section 4 analyzes the relationship between our criterion, envy-freeness, and the grading principle. Section 5 compares the degree of arbitrariness of the criterion with that of envy-freeness. Section 6, characterizes pseudo-egalitarian preference. Section 7 extends the analysis to $n$ agents, and Section 9 develops an additional example.

*Related literature:* Our work is primarily connected to the literature on interpersonal comparisons which, in response to the skepticism of Robbins (1935), seeks to enrich the informational basis of social choice (Sen, 1976, 1977b, 1979; d'Aspremont and Gevers, 1977).

More precisely, we adopt a methodology based on extended sympathy, a concept introduced by Arrow (1963) and Arrow (1977). As noted above, two main criteria have been proposed. First introduced by Foley (1967) and subsequently related to Pareto efficiency by Varian (1974), envy-freeness has been extensively studied from an economic perspective (Feldman and Kirman, 1974; Thomson, 1982, 2011). It has also proved useful in mechanism design and matching theory (Fleurbaey and Maniquet, 1997).

The concept has played a significant role in political philosophy, where it underpins Dworkin (1981a,b)'s conception of equality as well as Kolm (1972)'s account of justice. More recently, it has been extensively studied in computational social choice and fair division (Budish, 2011; Lipton et al., 2004; Segal-Halevi and Sziklai, 2019; Caragiannis et al., 2019).

The grading principle, first introduced by Suppes (1966) and later developed by Sen (1970), has recently found renewed applications in intertemporal social choice theory (Asheim et al., 2010). In particular, it has been applied to growth models (Banerjee, 2017).

Part of the literature has focused on whether Arrow's impossibility results survive when such informational enrichments are incorporated (Roberts, 1979, 1980; Suzumura, 1996). Our objective, by contrast, is to identify a class of social welfare functions satisfying Consensus on the Lesser Evil.

Several candidates for measuring the welfare that an individual with preference $\succsim_i$ derives from an allocation $f_i$, and thereby enabling such comparisons, have been proposed in the literature. Pazner (1978) applies a max–min criterion to a score by a reference bundle.

Dhillon and Mertens (1999), Karni (1998), and Segal (2006) provide axiomatizations of relative utilitarianism, which consists in maximizing the sum of suitably normalized utilities.

In their book, Fleurbaey and Maniquet (2011) develop the Egalitarian Walrasian ordering, in which the welfare of $f_i$ is measured by the minimal expenditure required to attain a bundle at least as good as $f_i$, for prices chosen to maximize the welfare of the least advantaged agent. Finally, Piacquadio (2017) characterizes opportunity-equivalent well-being functions relative to an opportunity set.

Our criterion is studied under leximin preferences, which we also characterize drawing on Hammond (1976). For its interpretation, we rely on veil-of-ignorance arguments in the spirit of Harsanyi (1955)'s justification of utilitarianism.

# 1 Framework

In this section, we introduce the framework and the class of pseudo-egalitarian preferences, represented through the leximin criterion. We then show that such preferences can be almost dictatorial—systematically favoring one agent over the other—which demonstrates that additional ethical constraints are called for.

## 1.1 Individual Preferences

Let $\Omega$ be a set of states. Let $F \equiv [0,1]^{\Omega}$ be the space of acts à la Savage, and write 0 and 1 for the constant acts. A feasible allocation is a pair $f = (f_1, f_2) \in \mathcal{F}$, where $f_i \in F$ is the act assigned to individual $i$. We assume throughout that $\mathcal{F} \subseteq F^2$ is compact and convex and that $(1,0), (0,1) \in \mathcal{F}$.[3]

A common special case is statewise resource feasibility,[4]

$$\mathcal{F} \equiv \{(f_1, f_2) \in F^2 : f_1(\omega) + f_2(\omega) \leq 1 \text{ for all } \omega \in \Omega\}.$$

Each individual $i \in \{1,2\}$ has a preference relation $\succeq_i$ on $F$, with associated strict preference $\succ_i$ and indifference $\sim_i$, assumed to be a continuous and monotonic weak order. For $f, g \in \mathcal{F}$, say that $f$ dominates $g$ if each individual weakly prefers her own component under $f$ and at least one individual strictly prefers it. The efficient frontier is then

$$E \equiv \{f \in \mathcal{F} : \text{no } g \in \mathcal{F} \text{ dominates } f\}.$$

Later sections compare the disadvantaged positions generated by efficient allocations, so we also record the coordinate projections

$$E_i \equiv \{f_i \in F : \exists f_{-i} \in F \text{ with } (f_i, f_{-i}) \in E\}, \qquad i \in \{1,2\}.$$

We first assume that preferences are such that the Pareto set is continuous. This assumption is made for convenience and could be avoided at the cost of adding conceptually uninformative complications.

**Assumption 1** (Pareto Continuity)**.** *For all $i \in \{1,2\}$ and all $f_i \in F$, there exists $g_i \in E_i$ such that $g_i \sim_i f_i$.*

More substantively, we also require that preferences be co-monotonic on the Pareto set in the following sense.

[3] The rich act space is a modeling device rather than a commitment to physical divisibility. In applications with indivisible objects, intermediate acts can be read as assignment probabilities or as counterfactual benchmarks used to formulate fairness restrictions. The social ranking is characterized on the rich domain and can then be restricted to the subset of policies treated as admissible in the application.

[4] This specification covers state-dependent budget shares, state-dependent commodity allocations, and probabilistic assignments of indivisible opportunities. In such applications, $f_i(\omega)$ can be read as the amount, share, or assignment probability delivered to individual $i$ in state $\omega$, while heterogeneity in needs, risk attitudes, outside options, or state-dependent valuations is absorbed by the individual's ordinal preference over acts.

**Assumption 2** (Pareto Comonotonicity). *For all $i \in \{1,2\}$ and all $f_i, g_i \in E_i$,*

$$f_i \succeq_i g_i \quad \textit{if and only if} \quad f_i \succeq_{-i} g_i.$$

This assumption can equivalently be taken as an axiom on individual preferences, since our main representation both implies and requires it. It is satisfied whenever the Pareto set is monotonic, as individual preferences are then aligned along it.

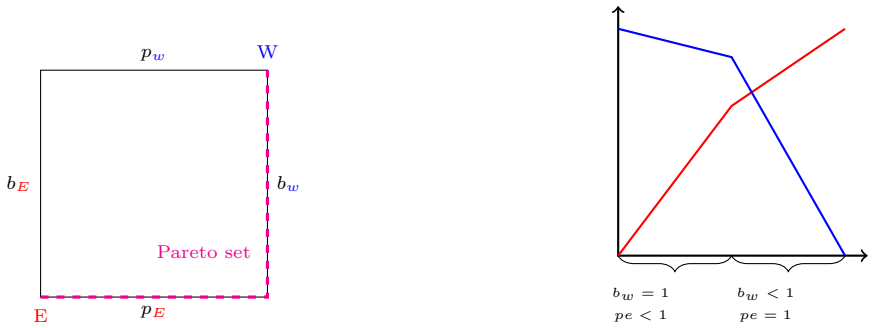
The framework is intentionally agnostic about the individual's model of choice under risk or uncertainty because $\succeq_i$ is primitive. Expected-utility preferences in the sense of Anscombe and Aumann (1963), rank-dependent preferences (Quiggin, 1982; Wakker, 2010), maxmin expected utility (Gilboa and Schmeidler, 1989), and Choquet expected utility (Schmeidler, 1989) all fit as long as they induce continuous monotone ordinal rankings on acts. The fairness restrictions studied below are therefore imposed on the social ranking rather than on a specific parametric model of individual behavior.

**Berlin Reunification** *Berlin's municipal reunification provides a useful illustration because it combined distributive and institutional conflict across several dimensions at once. In 1990 the West Berlin Senate and the East Berlin Magistrat briefly governed through the joint "MagiSenat," and the first unified election followed in December 1990 (Der Regierende Bürgermeister von Berlin – Senatskanzlei, nd; Der Landeswahlleiter Berlin, 1990). Official chronologies emphasize rapid political reintegration (Bundesregierung, nd), while the historical literature documents conflict over policing, university restructuring, and urban rebuilding priorities (Polizei Berlin, nd; Einstein Forum, nd; Loeb, 2006; Urban, 2007; Schwedler, 2001).*

*We model this East and West by assuming linear utilities*

$$u_e(b,p) = \beta_e b + \rho_e p, \qquad u_w(b,p) = \beta_w b + \rho_w p, \qquad \frac{\rho_e}{\beta_e} > \frac{\rho_w}{\beta_w}.$$

*The inequality indicates that East values exert greater control relative to the budget than West values. As illustrated in Figure (a), the Pareto set (efficient frontier) is L-shaped within the Edgeworth box (see also Appendix H). Figure (b) plots the utilities of both agents along the Pareto set.*




(a) Edgeworth box. (b) Utilities over $E$.

Figure: Allocation space and utility representation.

□

## 1.2 Pseudo-egalitarian Social Preferences

A social preference is a weak order $\succeq$ on $\mathcal{F}$. Since $\Omega$ is finite, $\mathcal{F}$ inherits the Euclidean topology from $F^2$. We focus on *pseudo-egalitarian* social preferences, which can be represented by the Leximin order. The Leximin is defined, as usual, as an order on $\mathbb{R}^2$ is such that for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$

$$\mathbf{x} \geq_{\text{Lex}} \mathbf{y} \iff \min(x_1,x_2) > \min(y_1,y_2) \text{ or } \left\{ \begin{array}{c} \min(x_1,x_2) = \min(y_1,y_2) \\ \max(x_1,x_2) \geq \max(x_1,x_2)) \end{array} \right\}.$$

The leximin order of a given pseudo-egalitarian social preference will apply to a *metric*, which we define as a pair $\mathbf{m} = (m_1, m_2)$ such that for each $i \in \{1, 2\}$, $m_i : F \to \mathbb{R}$ represents $\succeq_i$ and,

$$m_1(0) = m_2(0) \qquad \text{and} \qquad m_1(1) = m_2(1).$$

We refer to $m_i$ as the *metric score of* $i$; and for any function $\phi : \mathbb{R} \mapsto \mathbb{R}$ and any metric $\mathbf{m} = (m_1, m_2)$ we write $\phi \circ \mathbf{m} \equiv (\phi \circ m_1, \phi \circ m_2)$.

**Definition 1.** *A social preference is* pseudo-egalitarian *is there exists a metric* $\mathbf{m}$ *such that if for all* $f, g \in \mathcal{F}$*,*

$$f \succeq g \quad \iff \quad \mathbf{m}(f) \geq_{\text{Lex}} \mathbf{m}(g), \qquad \mathbf{m}(f) \equiv (m_1(f_1), m_2(f_2)).$$

Pseudo-egalitarian social preferences can be characterized by four axioms: Pareto, Hammond's equity axiom, as well as a scaling and a decomposition axiom. Since this characterization is not central to the main argument, we defer the formal statement and proof to Section 6.

For the purposes of the present analysis, it is useful to highlight two key properties of pseudo-egalitarian preferences. First, any metric $\mathbf{m}$ representing such preferences is unique up to an increasing homeomorphism from $[0, 1]$ to $[0, 1]$. Second, a pseudo-egalitarian social preference admits a unique element $f^{\succeq} \in E$ that maximizes $\succeq$ over the Pareto set.

**Proposition 1.** *For any pseudo-egalitarian preference* $\succeq$ *the two following statements hold*

*(i) if* $\mathbf{m}$ *and* $\mathbf{m}'$ *represent* $\succeq$ *if and only if there exists any increasing homeomorphism* $\phi$ *from [0,1] to [0,1] such that* $\mathbf{m}' = \phi \circ \mathbf{m} \equiv (\phi \circ m_1, \phi \circ m_2)$*;*

*(ii) there exists a single* $f^{\succeq} \in E$ *such that* $f^{\succeq} \in \max(E; \succeq)$*.*

Property (i) establishes that, under pseudo-egalitarian preferences, fixing a score metric for one agent uniquely determines the metric for the other agent. Accordingly, while each metric is unique up to an increasing transformation, the pair of score functions must satisfy a consistency requirement.

Property (ii) shows that a pseudo-egalitarian preference admits a unique maximizing element. Crucially, $f^{\succeq}$ may lie anywhere on the interior of the Pareto set, thereby favoring one agent over the other. A pseudo-egalitarian social preference may thus arbitrarily prioritize one agent over the other by freely determining who is treated as the worst-off. Based on this fact, the next proposition shows that the concept of pseudo-egalitarian social preference is nearly compatible with dictatorial social preferences, in the sense that a pseudo-egalitarian preference can always be constructed to be arbitrarily close to a dictatorial preference. The rest of the paper therefore does not revisit the leximin form established here. It studies which additional restrictions can narrow the remaining normalization freedom. Let us illustrates for now what a pseudo-egalitarian preference requires and why it can be almost dictatorial.

**Proposition 2** (Arbitrariness within pseudo-egalitarianism)**.** *For any* $i \in \{1, 2\}$*, consider the* $i$*-dictatorial* $\succeq^*$ *social preference such that*

$$f \succeq^* g \quad \iff \quad f_i \succeq_i g_i.$$

*There exists a pseudo-egalitarian preference as closed as we wish to* $\succeq^*$*, i.e. for any neighborhood* $V$ *of* $\succeq^*$*, there exists a pseudo-egalitarian social preferences*[5] $\succeq \in V$*.*

This confirms that pseudo-egalitarian preferences are grounded in insufficient ethical content, thus calling for the introduction of additional criteria. Next section is devoted to that task.

[5]The result is built upon the topology on social preferences introduced in Appendix A, where the proof also relates it to standard product topologies on binary relations.

**Berlin Reunification (Continued)** *On the Pareto set, a pseudo-egalitarian social preference $\succeq$ determines which agent is prioritized through the maximal element $f^{\succeq}$. If the social planner considers that East Berlin is worse off under an efficient allocation $f \in E$ than under the maximizing allocation $f^{\succeq}$—that is, if $f_e^{\succeq} \succ_e f_e$—then it must be that*

$$m_w(f_w) \geq m_w(f_w^{\succeq}) = m_e(f_e^{\succeq}) \geq m_e(f_e),$$

*and therefore East Berlin is regarded as the worst-off agent under this allocation.*

*Figure (a) illustrates this case for a social preference based on the original utilities of East and West Berlin. On the Pareto set, the welfare function—which therefore maximizes only the position of the worst-off agent—is represented by the thick curves, whose maximum occurs at the point where both utilities are equalized. Figure (b) depicts the welfare function corresponding to a different social preference, in which utilities have been rescaled so that the maximum occurs at the allocation where East Berlin receives all power and West Berlin receives all the budget.*

*To build intuition for Proposition 2, suppose the social preference follows the leximin ordering of the transformed utilities*

$$\left(\frac{1+u_e}{2}, \frac{u_w}{2}\right).$$

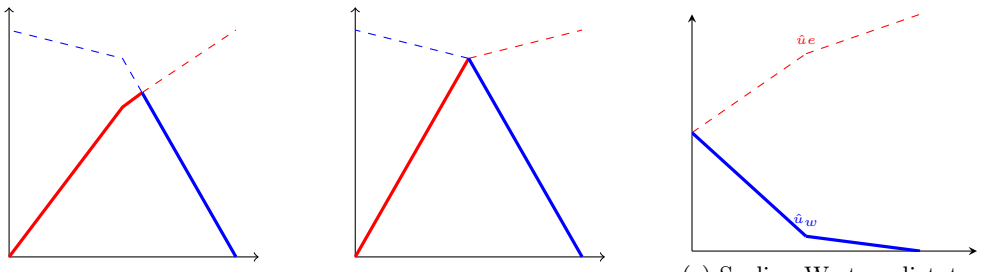
*For any $(p_e, b_e) \in E_e$ and $(p_w, b_w) \in E_w$ we have*

$$\frac{1+u_e(p_e,b_e)}{2} \geq \frac{u_w(p_w,b_w)}{2}.$$

*Hence West Berlin is always perceived as the worst-off agent under this transformation. Consequently, the social preference relation $\succsim$, represented by $\min_i \hat{u}_i(f_i)$, is effectively determined by $\hat{u}_w(f_w)$. In other words, the social planner's ranking coincides with West Berlin's preferences, rendering West Berlin dictatorial in the social ordering. Note that such a social preference is not pseudo-egalitarian, since it is not normalized at the extreme allocations; that is, $m_i(0) = 0$ and $m_i(1) = 1$ do not hold.*

*To prove the proposition, one can therefore construct a sequence of utility pairs, each providing a leximin representation of some $\succeq$, which converges (in measure) to*

$$\left(\frac{1+u_e}{2}, \frac{u_w}{2}\right).$$




(a) Original Utilities. (b) Scaled Utilities. (c) Scaling West as dictator.

□

## 2 Consensus Lesser Evil

We can now describe the ethical principle we propose. To that purpose, consider two allocations $(f_1, 0)$ and $(0, g_2)$, each assigning the worst act 0 to one agent while allocating an efficient act to the other, with $f_1 \in E_1$ and $g_2 \in E_2$. Both allocations may be regarded as ethically undesirable. However, if one of them must be chosen, it may still be desirable to

rank them ethically in order to determine which constitutes the *lesser evil*. Our criterion requires that if there is a consensus, both agents agreeing to rank $f_1$ better than $g_2$, then the social preference must not go against this consensus and should rank $(f_1, 0)$ as socially better than $(0, g_2)$. In other word, the existence of consensus provides a mean of identifying the lesser evil.

Consensus Lesser Evil (CLE) *For distinct* $i, j \in \{1, 2\}$ *and all* $(f_i, g_j) \in E_i \times E_j$,

$$f_i \succeq_1 g_j \text{ and } f_i \succeq_2 g_j \quad \Longrightarrow \quad (f_i, 0) \succeq (0, g_j).$$

*If one of the two weak preferences is strict, then*

$$(f_i, 0) \succ (0, g_j).$$

To state the main representation, we need to introduce two classes of metrics. The first class is built by using one of the two agents as a benchmark, for constructing the metric score of the other agent. The second mixes the benchmarks of the two agents.

## 2.1 Benchmark based metrics

A natural type metric is one that uses one of the two agents, say agent $i$, as a benchmark. To build such metric one begins with a score metric a score metric for $i$, which can be whatever utility, normalized to the interval $[0, 1]$, and then defines the other agent's score metric by requiring the two scores to coincide on the set of acts that are efficient for the benchmark agent.

**Definition 2.** *A metric* $\mathbf{m}^i = (m_1^i, m_2^i)$ uses $i \in \{1, 2\}$ as a benchmark *if for all* $f_i \in E_i$, $m_1^i(f_i) = m_2^i(f_i)$.

With such a metric, the score assigned to any act $g_{-i}$ for non-benchmark agent $-i$, is therefore the score of the act that intersects the indifference curve of $-i$ and $E_i$. The score metric $m_{-i}^i$ is assumed to represent agent $-i$ and it need not coincide globally with $m_i^i$.

Notice that Assumptions 1 and 2 make this definition operative. Assumption 2 provides a necessary and sufficient condition for the metric to be well-defined. If this assumption is violated, any metric constructed in this way would fail to represent the preferences of both agents simultaneously. It can therefore be interpreted as an axiom underlying our representation, requiring a minimal degree of agreement between agents.

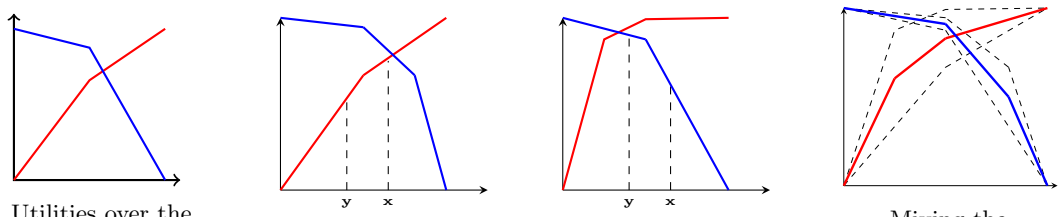
Assumption 1 ensures that such a metric is fully identified, in the sense that a score is assigned to every act. This is formalized in Proposition 3, which explicitly characterizes in (1) the unique pseudo-egalitarian social preference $\succeq^{b,i}$ represented by a metric that uses agent $i$ as a benchmark. This formulation follows directly from the equivalence between $(f_i, 0) \succeq^{b,i} (0, g_{-i})$ and

$$m_{-i}^i(f_i) = m_{-i}^i(f_i) \geq m_{-i}^i(g_{-i}),$$

where the first equality follows from Definition 2.

Two additional properties of $\succeq^{b,i}$ are worth emphasizing. First, although $\succeq^{b,i}$ is not dictatorial, it nonetheless tends to favor $i$, at least relative to $\succeq^{b,-i}$. This is formalized in Proposition 3.i., which states that whenever the social preferences $\succeq^{b,i}$ and $\succeq^{b,-i}$ disagree over a pair of allocations $(f, g)$, agent $i$'s preference between $f$ and $g$ aligns with $\succeq^{b,i}$, while agent $-i$'s preference aligns with $\succeq^{b,-i}$. Therefore, $\succeq^{b,i}$ tends to favor the satisfaction of agent $i$, while $\succeq^{b,-i}$ favors that of agent $-i$. This property is illustrated and discussed in Figure 1 in the following example.

Second, the allocation in $\mathcal{F}$ that maximizes $\succeq^{b,i}$ coincides with agent $i$'s most preferred envy-free allocation, and thus with agent $-i$'s least preferred one. This property, stated in Proposition 3.ii., is key to ensuring that mixed benchmarks remain compatible with envy-freeness.




Utilities over the Pareto set | $e$ as Benchmark | $w$ as Benchmark | Mixing the Benchmarks

Figure 1: Mixing Benchmarks from Original Utilities

**Proposition 3.** *Under Assumptions 1 and 2, the unique pseudo-egalitarian social, denoted by $\succeq^{b,i}$, that uses $i$ as benchmark satisfies*

$$\text{for all } (f_i, g_{-i}) \in E_i \times E_{-i}, (f_i, 0) \succeq^{b,i} (0, g_{-i}) \quad \text{if and only if} \quad f_i \succeq_{-i} g_{-i}. \tag{1}$$

*Moreover,*

*i. For any $f, g \in \mathcal{F}$, $[f \gtrsim^{b,i} g$ and $g \gtrsim^{b,-i} f]$ imply $[f_i \gtrsim_i g_i$ and $g_{-i} \gtrsim_{-i} f_{-i}]$.*

*ii. For each $i \in \{1,2\}$, $\max\left(\{f \in E : \forall j \in \{1,2\} f_j \succeq_j f_{-j}, \succ_i\right) = \max\left(E, \succeq^{b,i}\right)$*

**Berlin reunification (continued).** *Let us start by using the utility of East Berlin as the benchmark to define a common metric for both East and West Berlin: $m_e^e(p_e, b_e) = u_e(p_e, b_e)$. To construct the score metric of West Berlin, we then seek a continuous and increasing function $sc_e : [0,1] \to [0,1]$, which ensures that $sc_e \circ u_w$ represents $\succsim_w$, such that for every $f_e \in P_e$, $u_e(f_e) = sc_e\big(u_w(f_e)\big)$. The following piecewise linear function satisfies these properties:*[6]

$$sc_e(t) = \frac{\rho_e}{\rho_w} t \cdot \mathbb{1}_{t<\rho_w} + \left(\frac{\beta_e}{\beta_w} t + \rho_e - \frac{\beta_e \rho_w}{\beta_w}\right) \cdot \mathbb{1}_{t \geq \rho_w}$$

*Conversely, if we take West Berlin as the benchmark, we posit $m_w^w = u_w$, and use a scaling function $sc_w : [0,1] \to [0,1]$ such that $sc_w \circ u_e$ represents $\succsim_e$.*

$$sc_w(t) = \frac{\beta_w}{\beta_e} t \cdot \mathbb{1}_{t<\beta_e} + \left(\frac{\rho_w}{\rho_e} t + \beta_w - \frac{\rho_w \beta_e}{\rho_e}\right) \cdot \mathbb{1}_{t \geq \beta_e}$$

*then, the metric $\mathbf{m}^w = (m_e^w, m_w^w) = (sc_w \circ u_e, u_w)$, then uses Belin West as a benchmark.*

□

## 2.2 Mixed benchmarks

We have seen with Proposition 3.i., that a pseudo-egalitarian social preference using one agent as a benchmark tends to favor that agent. This naturally motivates the consideration of a mixed metric that aggregates two metrics, each taking a different agent as a benchmark.

**Definition 3.** *A metric $\mathbf{m} = (m_1, m_2)$ mixes benchmarks if there exist $\mathbf{m}^1$ and $\mathbf{m}^2$ respectively using 1 and 2 as benchmarks such that $\mathbf{m} = \frac{\mathbf{m}^1 + \mathbf{m}^2}{2}$.*

[6] First note that $u_w(p_e, b_e) = \rho_w p_e + \beta_w b_e < \rho_w$ if and only if $p_e < 1$. Hence

$$u_e(p_e, b_e) = \begin{cases} \rho_e\, p_e & \text{if } p_e < 1, \\ \rho_e + \beta_e b_e & \text{if } p_e \geq 1, \end{cases} = \begin{cases} \dfrac{\rho_e}{\rho_w}\, u_w(p_e, b_e) & \text{if } u_w(p_e, b_e) < \rho_w, \\ \dfrac{\beta_e}{\beta_w}\, u_w(p_e, b_e) + \rho_e - \dfrac{\beta_e \rho_w}{\beta_w} & \text{if } u_w(p_e, b_e) \geq \rho_w. \end{cases}$$

Therefore $u_e(pe, be) = sce(u_w(pe, be))$.

Our main finding is that CLE characterizes pseudo-egalitarian social preferences represented by such mixed benchmarks.

**Theorem 1.** *A social preference $\succeq$ is pseudo-egalitarian satisfies CLE if and only if it is represented by a metric $\mathbf{m} = (m_1, m_2)$ that* mixes benchmark. *We say that such social preference admits a mixed benchmark representation.*

Before interpreting this result, let us slightly extend it to include pure benchmark representations. Indeed, note that the representation of Theorem 1 excludes the case in which a single agent serves as the benchmark. This case can be incorporated by considering $\alpha$-mixed benchmark metrics.

**Definition 4.** *A metric $\mathbf{m}^\alpha = (m_1^\alpha, m_2^\alpha)$* $\alpha$-mixes-benchmarks *for some $\alpha \in [0,1]$ if there are two metrics $\mathbf{m}^1, \mathbf{m}^2$ taking respectively agent 1 and agent 2 as benchmark and*

$$m_j^\alpha = \alpha m_j^1 + (1-\alpha) m_j^2, \qquad j \in \{1,2\}.$$

The next corollary shows that social preferences represented by a metric that *$\alpha$-mixes-benchmarks* either satisfy CLE or belong to $\{\succeq^{b,1}, \succeq^{b,2}\}$.

**Corollary 1.** *A social preference $\succeq$ is pseudo-egalitarian and either it satisfies CLE or $\succeq \in \{\succeq^{b,1}, \succeq^{b,2}\}$ if and only if there exists $\alpha$ and a metric $\mathbf{m}^\alpha$ that $\alpha-$mixes benchmarks and represent $\succeq$.*

One may interpret the mixed representation as the social planner placing herself in the position of each agent in order to evaluate their joint welfare, and then combining these evaluations according to her own judgment. The planner's judgment matters because there remains an element of ethical underdetermination in how these evaluations should be aggregated. Indeed, for any increasing function $\phi : \mathbb{R} \to \mathbb{R}$, both $\mathbf{m}^1 = (m_1^1, m_2^1)$ and $\phi \circ \mathbf{m}^1 = (\phi \circ m_1^1, \phi \circ m_2^1)$ are benchmark metrics based on agent 1. Although these two metrics represent the same social preference, they assign different metric scores to both agent. As a result, mixing $\mathbf{m}^1$ with $\mathbf{m}^2$ generally does not yield the same mixed benchmark representation as mixing $\phi \circ \mathbf{m}^1$ with $\mathbf{m}^2$, unless the two agents have identical preferences. The social planner therefore retains discretion over how to weight or normalize each benchmark metric in light of her own evaluative judgment.

A complementary interpretation can be developed based on the notion of a dilemma, which has long been regarded as a central feature of "a central fact of ethical life."(see, e.g., Nagel (1979); Williams (1965)). Motivated by this perspective, we introduce the dilemma set:

$$\mathcal{D}^{\succeq} \equiv (f_1, g_2) \in E_1 \times E_2 : (f_1, 0) \sim (0, g_2).$$

A pair $(f_1, g_2) \in \mathcal{D}^{\succeq}$ is interpreted as representing a dilemma between the evil allocation $(f_1, 0)$ and the evil allocation $(0, g_2)$ because the social planner is indifferent between them, which we interpret as moral uncertainty. Using the contrapositive of CLE, the axiom can then be reformulated as:

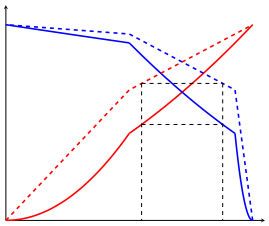
$$(f_1, g_2) \in \mathcal{D}^{\succeq} \implies \big(f_1 \succeq_1 g_2 \iff g_2 \succeq_2 f_1\big).$$

In words, CLE rules out false dilemmas in which both agents agree that one alternative is strictly better than the other. A key property of a representation based on the mixed benchmark $\mathbf{m}^1 + \mathbf{m}^2$ is that, for all $(f_1, g_2) \in \mathcal{D}^{\succeq}$,

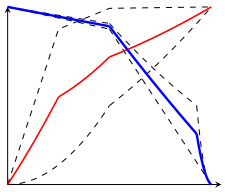
$$m_1^1(f_1) + m_1^2(f_1) = m_2^1(g_2) + m_2^2(g_2).$$

By Definition 2, this implies

$$(m_2^1 + m_1^2)(f_1) = (m_2^1 + m_1^2)(g_2) \quad \text{and} \quad m_1^2(f_1) - m_1^2(g_2) = m_2^1(g_2) - m_2^1(f_1). \tag{2}$$



$e$ as a benchmark under original utilities (dashed) and transformed utilities



Mixing the Benchmarks

Figure 2: Mixing benchmark after transforming East Berlin utility.

The first equality shows that the social planner evaluates both alternatives using a common utility index, $(m_2^1 + m_1^2)$, which must coincide across all dilemma pairs. Thus, any dilemma must be compatible with the existence of such a common utility, constructed by combining the score metrics of both agents.

The second equality clarifies why CLE is implied by a mixed representation. Indeed, if $f_1 \succeq_1 g_2$, this is equivalent to

$$m_1^2(f_1) - m_1^2(g_2) \geq 0,$$

which, by the above identity, is equivalent to

$$m_2^1(g_2) - m_2^1(f_1) \geq 0,$$

and hence to $g_2 \succeq_2 f_1$.

This also suggests that, from the perspective of the social planner, a dilemma equalizes the loss incurred by agent 1 when moving from $f_1$ to $g_2$ with the loss incurred by agent 2 when moving from $g_2$ to $f_1$.

**Berlin reunification (continued).** *The metric* $\mathbf{m}^e = (m_e^e, m_w^e) = (u_e, sc_e \circ u_w)$ *then takes East Berlin as a benchmark. The graph is depicted in the center panel of Figure 1: the score is left unchanged for East Berlin, while it is now scaled for West Berlin. Moreover, it is easy to verify that for any homeomorphism* $\phi$ *from [0,1] to [0,1], the metric* $\tilde{\mathbf{m}}^e = (\tilde{m}_e^e, \tilde{m}_w^e) = (\phi \circ u_e, \phi \circ sc_e \circ u_w)$ *also uses Est Berlin as a benchmark.*

$\mathbf{m}^E$ *defined above* $\mathbf{m}^W$ *identify two metrics that take both Berlin East and the West as a benchmark. A possible mixed benchmark is based on averaging* $\mathbf{m}^E$ $\mathbf{m}^W$*, which is depicted on the right panel of Figure 1. However, a mixed benchmark may also be based on* $\tilde{\mathbf{m}}^E$ $\mathbf{m}^W$*—depicted in Figure 2. While the* $\mathbf{m}^E$ *and* $\tilde{\mathbf{m}}^E$ *taken as the sole benchmark induce the same social preference, it is not the case of the two mixed benchmarks presented above.* □

## 2.3 Refinements

The representation obtained in Theorem 1 can be refined in two ways. First, CLE requires that if $(f_1, 0) \sim (0, g_2)$ for some $(f_1, g_2) \in E_1 \times E_2$, then $f_1 \succeq_1 g_2$ if and only if $f_1 \preceq_2 g_2$, which is compatible with $f_1 \preceq_1 g_2$ and $f_1 \succeq_2 g_2$. One may instead require the stronger requirement that $(f_1, 0) \sim (0, g_2)$ implies $f_1 \succeq_1 g_2$ and $f_1 \preceq_2 g_2$.

Strong-CLE: if $(f_1, 0) \sim (0, g_2)$ for some $(f_1, g_2) \in E_1 \times E_2$, then $f_1 \succeq_1 g_2$ and $g_2 \succeq_2 f_1$.

From this stronger axiom we obtain a similar representation, that yet further constraint that the benchmark score of an agent to be lower than that of the score of the other agent under the same benchmark, i.e. $m_{-i}^i(f_i) \geq m_i^i(f_i)$ for all $f_i \in F$.

**Theorem 2.** *A pseudo-egalitarian social preference satisfies Strong-CLE if and only if there exists a mixed metric that represents it such that* $m_{-i}^i(f_i) \geq m_i^i(f_i)$ *for all* $f_i \in F$.

# 3 Implications for welfare evaluation in practice

This section clarifies what the theory offers for applied welfare evaluation. The goal is not to pin down a complete welfare, but to provide a principled way of ruling out unfair social rankings. Our approach is thus broadly constitutionalist in spirit: CLE first supplies the constitutional constraint that excludes social preferences which can be deemed unfair ex ante, and the policymaker then chooses among the options that survive this ethical filter. Three practical examples can illustrate this.

## 3.1 "Égalité devant l'impot"

Taxation is a natural testing ground for the framework, precisely because constitutional constraints impose ethical requirements on tax policy without resolving how they should be operationalized. Article 13 of the 1789 Declaration requires public contributions to be distributed in proportion to ability to pay (République française, 1789). As legal scholarship emphasizes, this principle constrains tax policy without uniquely determining how equality of contribution should be interpreted in practice (Ayrault, 2020). The difficulty is not merely technical: it reflects genuine disagreement about which indicator best captures who is disadvantaged by a given tax schedule, and therefore about which comparisons are normatively legitimate.

The recent debate over the so-called "Taxe Zucman" illustrates this starkly. Gabriel Zucman argued that a minimum tax on the top 1% was not merely desirable but constitutionally required, on the grounds that effective tax rates become regressive at the top of the wealth distribution—meaning that the ultra-wealthy contribute a smaller share of their resources than those below them (Bach et al., 2023). On this reading, the relevant comparison basis is the effective tax rate as a function of wealth, and once that basis is accepted, the constitutional principle of proportionality unambiguously condemns the status quo. Many opponents, however, disputed precisely this: not the constitutional principle itself, but the claim that effective tax rates on wealth constitute the appropriate indicator of fiscal burden. Critics pointed to alternative bases—total lifetime tax contributions, consumption-adjusted income, or the role of capital taxation in financing public goods—under which the existing schedule appears far less regressive, or even progressive. The constitutional constraint was therefore shared; what was contested was the comparison basis through which it should be assessed.

This is exactly the structure the framework is designed to illuminate. CLE does not adjudicate between competing indicators of disadvantage. Rather, it clarifies what follows once a publicly defensible basis is accepted: if all agents agree—evaluated through that basis—that one efficient reform places the disadvantaged in a less severe position than another, indifference between them is difficult to justify. When the basis itself is contested, as in the Zucman debate, the framework predicts persistent disagreement rather than a determinate ranking, and treats this incompleteness as an honest reflection of the underlying normative dispute rather than a failure of the method. This perspective resonates with modern optimal-tax analysis when it makes the normative role of social preferences explicit (Saez and Stantcheva, 2016), and with evidence showing that the public acceptability of a tax yardstick is itself part of the evaluative problem (Stantcheva, 2021).

## 3.2 Retirement reform

Retirement-age reform offers a complementary illustration, and one that is particularly concrete in the French context. The 2023 pension reform, which raised the legal retirement age from 62 to 64, met with sustained and unusually broad opposition. The debate was not primarily about fiscal sustainability—that constraint was largely shared—but about whether a uniform increase in the retirement age constitutes an equal burden across the population. The ethical fault line lay elsewhere: in disagreement about the appropriate basis for comparing who is disadvantaged by the reform.

The case for a shared comparison basis is compelling here. Longevity and healthy-life expectancy vary systematically across socioeconomic groups: manual workers, those in physically demanding or hazardous occupations, and individuals with lower lifetime incomes retire in worse health on average, and die sooner after retirement (Fleurbaey and Schokkaert, 2013). On this basis, a uniform increase in the retirement age is not neutral—it compresses the retirement period precisely for those who have contributed the most physically and who have the least time to enjoy it. Health and longevity measures thus emerge as natural candidates for a publicly shared comparison basis on the efficient set of feasible pension schedules. They do not claim full interpersonal comparability across all dimensions of life; rather, they identify a dimension that is normatively salient for evaluating the burden induced by reform.

Yet the basis was actively contested. Opponents of using health and longevity as the primary indicator pointed to other dimensions that point in different directions: lifetime income trajectories, access to early retirement provisions already built into the system, caregiving burdens concentrated among women, and differences in job conditions that are imperfectly captured by occupational categories alone. Proponents of the reform, meanwhile, emphasized aggregate pension system sustainability and intergenerational equity as the relevant frame, under which the burden falls not on current retirees but on younger cohorts facing longer contribution periods. These are not disagreements about facts but about which dimension legitimately grounds the comparison.

CLE structures this debate without resolving it by fiat. When a basis commands broad acceptance—as health and longevity plausibly do for occupational categories with documented differential mortality—it implies that indifference between efficient reforms that differentially burden those groups is difficult to sustain. When the basis is contested, as it is once caregiving, lifetime income, and intergenerational considerations enter, the framework honestly predicts persistent dilemmas rather than forcing artificial precision. This interpretation sits comfortably alongside responsibility-sensitive welfare analysis (Fleurbaey and Schokkaert, 2013) and health-ethics approaches that emphasize publicly defensible priority principles under disagreement (Daniels, 2008).

### 3.3 Targeting and Eligibility Rules

Targeting and eligibility rules illustrate the same logic in a setting that is, in some respects, simpler. Many programs—means-tested benefits, disability insurance, housing assistance, or age-based entitlements—rely on publicly recognized thresholds to identify who counts as disadvantaged: poverty lines, disability scores, age cutoffs, residency criteria. These thresholds do not emerge from a complete theory of disadvantage; they are, precisely, publicly defensible indicators that society has agreed to treat as salient for the purpose of a specific program. As such, they are natural candidates for the comparison basis in CLE.

The contribution of the framework here is to clarify what these thresholds legitimately imply. When two efficient policy options differ in how they treat individuals on either side of a recognized threshold, and when all affected parties agree—evaluated through that threshold—that one reform places the eligible group in a less severe position, continued indifference between the two reforms is difficult to justify. The threshold does not merely label who is disadvantaged; it disciplines the ranking of efficient policies in a way that purely ordinal analysis cannot.

Yet the limitations of threshold-based reasoning are equally important to acknowledge, and the framework does not paper over them. Hard cases are endemic: near eligibility cutoffs, small differences in observable characteristics generate discontinuous treatment, raising questions about whether the threshold is tracking disadvantage or merely administering it. Under local variation in needs, prices, or living costs, a uniform poverty line may misclassify systematically. With multidimensional deprivation—where income, health, housing, and social participation point in different directions—no single threshold commands uncontested authority. And whenever the indicator is itself politically unstable or technically contested, the comparison basis loses the public acceptance that gives it normative force.

In these cases, CLE does not dissolve the difficulty by construction. Instead, it reframes it honestly: residual incompleteness in the ranking of efficient policies reflects genuine dis-

agreement about the appropriate comparison basis, not a defect of the method. The framework clarifies when a threshold legitimately constrains evaluation and when the demand for a determinate ranking outruns what publicly defensible indicators can deliver.

### 3.4 What the Framework Delivers

The three illustrations share a common structure, and it is worth stating explicitly what the framework delivers across all of them. In each case, a constitutional or ethical principle commands broad agreement—proportionality of fiscal burden, equitable distribution of reform costs, recognition of publicly identified disadvantage—yet underdetermines the ranking of efficient policies because it does not specify which indicator should operationalize it. CLE does not resolve this underdetermination by imposing a global interpersonal metric. It asks only whether two efficient compromises that favor different groups generate disadvantaged positions that all affected parties rank in the same direction. When they do, continued social indifference is difficult to justify. When they do not, the resulting incompleteness honestly reflects either genuine disagreement about disadvantage or disagreement about which public indicator should operationalize it—and in pluralistic settings, this is often the right outcome rather than an embarrassment to be eliminated by construction (Sen, 2017).

Theorem 1 makes this precise by characterizing the full class of metrics compatible with CLE. The structure it reveals is striking: a publicly defensible indicator qualifies if and only if it satisfies a symmetry condition whereby each agent's self-evaluation coincides with the evaluation the other agent assigns to them at every efficient allocation. This impartial-spectator requirement—aligning how one assesses one's own situation with how an outside observer would assess it—is far from obvious from the definition of CLE alone, and it delivers a concrete test for applied work. An indicator passes if it would be endorsed not only by those whose disadvantage it measures but also by those evaluating that disadvantage from the outside. Indicators that fail this test cannot support a CLE-consistent ranking, and the framework correctly leaves the corresponding comparisons unresolved rather than forcing a conclusion.

Envy-freeness offers a different route to the same baseline. It treats swap-based objections as decisive on the efficient set—an appealing approach because such objections are formulated in within-person terms and, in divisible environments, are closely connected to competitive-equilibrium reasoning (Foley, 1967; Varian, 1974). The two criteria are not rivals but complements: one gives decisive weight to the no-envy boundary, the other to shared judgments about which disadvantaged position is less severe, disciplined by the symmetry condition Theorem 1 identifies. Both refine the same pseudo-egalitarian baseline in ways that purely ordinal analysis cannot. The practical contribution of the paper is therefore not to recommend a privileged statistic for taxation, pensions, or targeting, but to provide a disciplined method for incorporating publicly defensible indicators without treating them as already implied by ordinal information alone—drawing a clear line between conclusions that follow logically from shared ethical commitments and those that depend on contestable judgments about the appropriate comparison basis.

## 4 Relation to existing fairness concept in the literature

In this section, we relate our ethical criterion to two prominent extended-sympathy-based ordinal criteria: the *Grading principle* and *Envy-freeness*. We show that CLE is closely related to a weak version of the grading principle introduced by Suppes (1966), and can be interpreted as extending its logic.

We then establish that, although CLE is logically distinct from envy-freeness, it can be made compatible with it. We conclude that our criterion provides a natural complement to existing criteria in the literature, without generating impossibility when combined with them.

## 4.1 Grading principle

Suppes (1966) introduces the grading principle as an ordering $J_i$ over allocations that ranks them according to how just they are for agent $i$. Under this ordering, an allocation $f$ is judged more just than $g$ if either agent $i$ considers both individuals at least as well off in $f$ as in $g$, or if she considers both individuals in $f$ at least as well off as the other individual in[7] $g$. Formally:

**Definition 5.** *$fJ_ig$ if either $[f_1 \succsim_i g_1$ and $f_2 \succsim_i g_2]$ or $[f_1 \succsim_i g_2$ and $f_2 \succsim_i g_1]$.*

Suppes (1966, Theorem 2) shows that this relation is transitive but incomplete. It is, however, well known that $J_i$ may conflict with Pareto efficiency requirements[8]. To avoid this difficulty, we restrict the principle to Pareto-efficient allocations, as captured by the following axiom.

Grading through $i$ ($\mathrm{GT}_i$). For all $f, g \in E$, if $fJ_ig$, then $f \succsim g$.

As d'Aspremont (1985, p. 6) notes, the grading principle "amounts to making interpersonal comparisons from the point of view of a single person." Adopting such a viewpoint requires either assigning a special status to that individual (e.g., as an ethical observer) or, more fundamentally, providing a way to identify and reconcile the viewpoints of all individuals, each treated as a distinct ethical observer.

It is therefore natural to ask whether $\mathrm{GT}_i$ can be related to score-based metrics that take a particular agent as a benchmark, as in Definition 2. The next proposition establishes a close—yet perhaps surprising—connection: a social preference satisfying $\mathrm{GT}_i$ is represented by a metric that uses $-i$ (rather than $i$) as the benchmark.

**Proposition 4.** *A pseudo-egalitarian social preference satisfies $\mathrm{GT}_i$ if and only if it is represented by $\mathbf{m}^{-i}$.*

In light of Proposition 3.i., this implies that applying the grading principle from the perspective of agent $i$ leads to favoring agent $-i$. This observation naturally suggests combining $J_1$ and $J_2$ by establishing a connection between CLE and the $J_i$. In particular, a natural requirement is that whenever both agents deem $(f_1, 0)$ more just than $(0, g_2)$, the social preference should reflect this judgment.

Mixed Grading (MG). If $(f_1, 0)J_i(0, g_2)$ for each $i \in \{1, 2\}$, then $(f_1, 0) \succsim (0, g_2)$, with strict preference if not both $(g_2, 0)J_i(0, f_1)$ hold.

The next proposition establishes the equivalence between MG and CLE.

**Proposition 5.** *If $\succsim$ is pseudo-egalitarian, then MG is equivalent to CLE.*

This result shows that our criterion extends the logic of the grading principle. When restricted to Pareto-efficient allocations, $\mathrm{GT}_i$ induces a $-i$ benchmark metric; by linking the $J_i$ through MG, CLE avoids the need to privilege the viewpoint of a single agent while preserving the underlying comparative logic.

## 4.2 Envy-freeness

Envy-freeness is among the most prominent extended-sympathy–based criteria for comparing individuals' positions. It requires that, among two Pareto-efficient allocations, the social preference select the envy-free allocation—i.e., the one in which each agent prefers her own bundle to that of the other—whenever this property is satisfied by only one allocation.

First introduced by Foley (1967) and later linked to efficiency and competitive equilibrium by Varian (1974, 1976), we adopt this requirement here.[9] As emphasized by Thomson

[7] Suppes defines this ordering as asymmetric. For our purposes, it is both simpler and more convenient to work with a weak order that allows for indifference.

[8] See Sen (2017, Chapter 9 and 9$^\star$) or Suzumura (1996, Theorem 4).

[9] In Varian (1974, 1976)'s terminology, an allocation is *fair* if it is both Pareto efficient and envy-free. We retain this subordination to efficiency and focus on allocations satisfying both properties.

(1982), this criterion is informationally parsimonious: it requires only information about how individuals rank bundles within a given allocation.

Envy-freeness also plays a central role in the philosophical literature on distributive justice, most notably in Dworkin (1981a,b). In that framework, the envy test is used to characterize fair allocations and underpins the development of the luck egalitarian perspective. Crucially, this use of the envy test provides the foundation for luck egalitarianism. By imagining allocations emerging from a hypothetical equal starting point—such as Dworkin's auction—envy-freeness ensures that any resulting inequalities reflect differences in individual choices rather than differences in initial circumstances. In this sense, the absence of envy can be understood as a behavioral manifestation of responsibility-sensitive fairness: individuals do not object to others' outcomes when these outcomes arise from voluntary decisions under equal conditions.[10]

Given the appeal of envy-freeness and this importance of this criterion in the literature, it is therefore crucial to ask about the compatibility of CLE with Envy-freeness. Let introduce the set of envy-free (and efficient) allocations:

$$EF = \{f \in E : f_i \succeq_i f_{-i}, \forall i \in \{1,2\}\}.$$

Given the continuity of the $\succeq_i$, this set is compact. Moreover, since it is contained in $E$, it follows that $EF$ is a one-dimensional manifold embedded in $E$, with two extreme points $f, g \in E$ such that $f_1 \sim_1 f_2$ and $g_2 \sim_2 g_1$. The former corresponds to the envy-free allocation most favorable to agent 2, while the latter is the envy-free allocation most favorable to agent 1. The envy-freeness axiom can then be formulated as follows.

Envy-freeness (EF) For all $f \in EF$ and all $g \in E \setminus EF$, $f \succ g$.

In the context of pseudo-egalitarian social preference this criterion reduces to a very limited requirement, as shown by the following Proposition.

**Proposition 6.** *If $\succeq$ is pseudo egalitarian, then it satisfies EF if and only if for all $f, g \in E$*

$$\left\{ \begin{array}{l} f_1 \sim_1 f_2 \\ g_2 \sim_2 g_2 \end{array} \right\} \implies (f_1, g_2) \in \mathcal{D}^{\succeq}. \tag{3}$$

Proposition 6 shows that, in the context of pseudo-egalitarian preferences, envy-freeness imposes only a boundary condition. Specifically, it requires that the two extreme points of the envy-free set—namely, the allocations $f, g \in E$ such that $f_1 \sim_1 f_2$ and $g_2 \sim_2 g_1$—constitute dilemmas for the social planner. Condition (3), and thus EF, need not be satisfied by a social preference satisfying CLE. To see why consider the social preference $\succeq$ with the mixed-benchmark representation of

$$\mathbf{m} = \frac{\mathbf{m}^1 + \mathbf{m}^2}{2}$$

where the extreme points EF the social planner is indifferent between $f, g$. Hence,

$$\frac{m_1^1(f_1) + m_1^2(f_1)}{2} = \frac{m_2^1(g_2) + m_2^2(g_2)}{2}.$$

Notice that the social preference $\succeq'$ represented by

$$\mathbf{m} = \frac{\alpha \cdot \mathbf{m}^1 + (1-\alpha) \cdot \mathbf{m}^2}{2} \tag{4}$$

[10]This interpretation has been further developed and critically examined in subsequent philosophical work. In particular, Cohen (1989) emphasizes the distinction between brute luck and option luck, clarifying the normative foundations of responsibility-sensitive egalitarianism. Arnsperger and Van Parijs (2004) and Roemer (1998) provide formal and economic articulations of these ideas, while Kolm (1996) offers a broader normative framework connecting envy-freeness to notions of equal freedom and justice.

for some $\alpha \in (0,1)$ also satisfies CLE since (4) mixes benchmark. But if $f \neq g$ (i.e. $\succeq_1 \neq \succeq_2$), then

$$\frac{\alpha \cdot m_1^1(f_1) + (1-\alpha) \cdot m_1^2(f_1)}{2} \neq \frac{\alpha \cdot m_2^1(g_2) + (1-\alpha) \cdot m_2^2(g_2)}{2}.$$

violating condition (3). Moreover, condition (3) is too weak to imply CLE. The reason is that it leaves unrestricted the ranking of allocations $f, g \notin EF \setminus 0, 1$, and may therefore rank one allocation above another even when both agents agree that the least advantaged individual in the former is worse off than the least advantaged individual in the latter. Together with the previous argument, this establishes the logical independence of CLE and envy-freeness: neither criterion implies the other.

However, for any preference profile, there exists a social preference satisfying CLE that also satisfies (3), and hence envy-freeness. This follows from the fact that any pseudo-egalitarian preference admitting a mixed-benchmark representation has its maximal element $f^{\succeq}$ in the interior of $EF$. This result can even be made stronger: for any social preference $\succeq$ satisfying envy-freeness, there exists another social preference $\succeq'$ satisfying CLE such that $\succeq$ and $\succeq'$ coincide on the set $EF$. The following proposition summarizes these results.

**Proposition 7.** *If $\succeq$ is pseudo-egalitarian and satisfies EF, then there exists $\succeq'$ satisfying CLE and such that for any $f, g \in EF$*

$$f \succeq g \iff f \succeq' g.$$

To conclude, while EF and CLE are genuinely distinct criteria, in the context of pseudo-egalitarian social preferences they can always be made to operate in concert. In fact, CLE can replicate EF exactly on the set of envy-free allocations, suggesting that it provides a natural complement to EF, which is otherwise silent outside this set. This latter remark calls for an analysis of which criterion is less arbitrary than the other.

# 5 Comparing Envy-freeness and Consensus Lesser Evil

The goal of this section is to compare EF and CLE in terms of the degree of arbitrariness they permit, that is, the extent to which they allow one agent to be favored over the other. Several measures of arbitrariness could be considered for this purpose. The approach we adopt is best understood by analogy with a zero-sum game. First, an ethicist fixes an ethical criterion. Second, one of the two agents is designated as the social planner and is allowed to choose so as to favor herself. The ethicist then prefers criteria that minimize the extent to which the designated agent can bias outcomes in her favor, and such criteria are interpreted as less arbitrary. From this perspective, the ethical criterion acts as a constitutional constraint that limits the extent to which one part of the population can exploit the other, and the ethicist takes on the role of a constitutionalist.

Let $\mathcal{P}$ be the set of pseudo-egalitarian preferences. An ethical criterion is then a subset $\mathcal{P}$ generically denoted by $\mathcal{E}^{\cdot} \subseteq \mathcal{P}$, while $\mathcal{E}^{\mathcal{A}}$ stands for the set of pseudo-egalitarian preferences satisfying axiom $\mathcal{A}$. Let us introduce for each agent $i \in \{1,2\}$, define an order $\trianglerighteq_i \subseteq \mathcal{P}^2$ which replicates Proposition 3.i. and quantifies whether a social preference is *more favorable to $i$ on $A$*.

**Definition 6.** *For any $\succeq, \succeq' \in \mathcal{P}$, $\succeq$ is* more favorable than $\succeq'$ to $i$ on $A \subseteq \mathcal{F}$ *if for all $f, g \in A$*

$$\left\{ \begin{array}{c} f \succeq g \\ g \succeq' f \end{array} \right\} \implies f_i \succeq_i g_i.$$

*We then write $\succeq \rhd_i^A \succeq'$.*

In words, $\succeq$ is more favorable than $\succeq'$ to $i$ on $A \subseteq \mathcal{F}$ if it is more consistent with agent $i$'s preferences than $\succeq'$. It is straightforward to verify that the $\trianglerighteq_i^A$ is **transitive**, although

generally **incomplete**. Moreover, the next Proposition shows that $\langle \mathcal{E}, \trianglerighteq_i^A \rangle$ forms a complete lattice.

**Proposition 8.** *For each $i \in \{1,2\}$ each $\mathcal{E}$ and each $A$, $\langle \mathcal{E}, \trianglerighteq_i^A \rangle$ is a complete lattice, which therefore admits an infimum.*

With Proposition 8, we provide a definition to quantify the arbitrariness of ethical criteria. An ethical criterion $\mathcal{E} \subseteq \mathcal{P}$ can then be said to be more arbitrary than another $\mathcal{E}' \subseteq \mathcal{P}$ on $A$ if it offer more freedom to favor both agents.

**Definition 7.** *Consider two ethical criteria $\mathcal{E}, \mathcal{E}' \subseteq \mathcal{P}$, $\mathcal{E}$* is more arbitrary than *$\mathcal{E}'$ on $A$ if for both $i = 1, 2$,*

$$\inf(\mathcal{E}', \trianglerighteq_i^A) \trianglerighteq_i^A \inf(\mathcal{E}, \trianglerighteq_i^A).$$

In words, an ethical criterion is more arbitrary than another if it tends to disfavor more than the other the most disfavored agent. We are now in position to conduct our comparison exercise. We have seen with Proposition 3.i. that taking a an agent as a benchmark tends to favor him compare to taking the other agent as a benchmark.

In light of Theorem 1 and Corollary 1, this suggests that the ethical criterion $\mathcal{E} = \{\succeq^{b,1}, \succeq^{b,2}\}$ is more globally more arbitrary that $\mathcal{E}^{CLE}$, since the representation yielded by CLE mixes the benchmarks metrics obtained $\succeq^{b,1}, \succeq^{b,2}$. Such result however, can only be obtained under $sCLE$.

**Proposition 9.** *$\{\succeq^{b,1}, \succeq^{b,2}\}$ is more arbitrary than $\mathcal{E}^{sCLE}$ on $\mathcal{P}$. Moreover,*

$$\inf(\mathcal{E}, \trianglerighteq_1^{sCLE}) = \succeq^{b,2} \qquad \text{and} \qquad \inf(\mathcal{E}, \trianglerighteq_2^{sCLE}) = \succeq^{b,1}.$$

This result shows that comparing the arbitrariness of sCLE and EF amounts to comparing the arbitrariness of $\{\succeq^{b,1}, \succeq^{b,2}\}$ and $\mathcal{E}^{EF}$. For that reason we will focus on that comparison rather than the one between CLE and EF.

The next result shows that sCLE and EF cannot be ranked globally whenever both agents have the same preferences—an the envy-free set has an empty interior. It however, provides local comparisons: while sCLE is more arbitrary than EF on EF, the reverse holds outside EF.

**Proposition 10.** *If the envy-free set has an empty interior, then EF is globally more arbitrary than sCLE.*

*(a) If the envy-free set has a non-empty interior, neither is sCLE more arbitrary than EF, nor is EF on $\mathcal{P}$;*

*(b) sCLE is more arbitrary than EF on EF;*

*(c) EF is more arbitrary than sCLE on outside EF.*

While EF offers stronger safeguards against arbitrariness when envy-free allocations exist, the reverse holds when this is not the case: sCLE then provides stronger protection. In other words, sCLE provides a natural complement to EF, to deal with situations where not envyfree alocations can be achieved.

# 6 Axioms of pseudo-egalitarian Social Preference

This section characterize pseudo-egalitarian preferences in the environment of the paper. This characterization is based on four axioms.

**Axiom 1** (Pareto)**.** *If $f$ dominates $g$, then $f \succ g$.*

The second standing condition fixes a normalization at the two endpoint allocations. It does not impose anonymity on the whole ranking. It only rules out a label-based asymmetry at the endpoints and gives the later representation results a convenient reference point. It should be noted that it will be implied by our main axiom and it only serves to derives the Leximin representation.

**Axiom 2** (Symmetry (normalization))**.** $(1,0) \sim (0,1)$.

This section asks how much distributive structure is already fixed once the framework of Section 1 is combined with Pareto and a minimal ordinal equity discipline. Rawls's priority to the worse off suggests a leximin-type ranking (Rawls, 1971), but with only ordinal preference information that priority cannot be read from a given interpersonal scale. The section therefore starts from a weaker comparison device. It uses the planner's ranking of one-sided allocations in which one individual receives an act and the other is held at the common baseline 0. Those comparisons do not impose global interpersonal comparability. They only identify, in a limited and publicly observable way, which individual is socially disadvantaged across a pair of allocations. The result is pseudo-egalitarianism, an ordinal leximin structure with a precise residual normalization freedom. Later sections leave that priority form in place and study how additional restrictions such as CLE and envy-freeness can discipline the remaining freedom.

Write $-i$ for the individual other than $i$. For $f = (f_1, f_2) \in \mathcal{F}$ and $i \in \{1,2\}$, let

$$f^{i0} \equiv (f_i, 0) \qquad \text{and} \qquad f^{-i0} \equiv (0, f_{-i}).$$

These are the one-sided allocations associated with $f$. They isolate the planner's ranking of giving a particular act to one individual while the other remains at the common baseline. Because the argument runs through those one-sided comparisons, the section does not assume that the individuals' ordinal indices are already aligned on a common scale.

The first axiom is a coherence condition on social indifference. If two allocations generate the same one-sided tradeoff in both directions, then there is no remaining basis for ranking the full allocations differently. This is an ordinal separation restriction in the spirit of Hammond (1976).

**Axiom 3** (Decomposition (De))**.** *For any $f, g \in \mathcal{F}$ and $i \in \{1,2\}$, if*

$$f^{i0} \sim g^{-i0} \qquad \textit{and} \qquad g^{i0} \sim f^{-i0},$$

*then $f \sim g$.*

The second axiom carries the section's equity content. It applies only when the planner's one-sided ranking identifies the same individual as advantaged at both allocations. On that restricted domain, the social ranking must follow the disadvantaged individual's own ordinal ranking of what she receives. This is the minimal ordinal content behind maximin and leximin reasoning under limited comparability, and it is closest in spirit to the ordinal equity restrictions studied by Sen (1976), d'Aspremont and Gevers (1977), and Deschamps and Gevers (1978).

**Axiom 4** (Equity (Eq))**.** *For any $f, g \in \mathcal{F}$ and $i \in \{1,2\}$, if*

$$f^{i0} \succ f^{-i0} \qquad \textit{and} \qquad g^{i0} \succeq g^{-i0},$$

*then*

$$f \succeq g \iff f_{-i} \succeq_{-i} g_{-i}.$$

**Theorem 3** (pseudo-egalitarianism and aligned leximin)**.** *A social preference $\succeq$ satisfies Axioms 1-4 if and only if it is pseudo-egalitarian.*

Theorem 3 identifies the full distributive content of the pseudo-egalitarian baseline. Social evaluation first compares the worse represented position and uses the better one only to break ties. The endpoint equalities align the two ordinal scales at the common benchmark

acts 0 and 1, but nothing in the theorem selects a unique normalization away from those anchors. Pseudo-egalitarianism therefore fixes the priority form without fixing a unique interpersonal scale.

# 7 Discussion of the $n$ agents representation

Extending the logic of CLE to environments with more than two agents depends on the underlying notion of consensus. In particular, it is not clear whether consensus should involve all agents or only those affected by the comparison between two undesirable allocations. This distinction gives rise to two natural extensions of CLE.

To formalize this let us extend the framework to $n$ agents. Feasible allocations are now $n$ dimensional that belong to

$$\mathcal{F} = \{(f_1, ..., f_n) \in F^n : \sum_{k=1}^{n} f_k(\omega) \leq 1, \forall \omega \in \Omega\}.$$

The social preference is defined on $\mathcal{F}$. Let us denote for all $f_i \in F$ the allocation $f_i^{|j}$ the allocation which allocates $f_i$ to agent $j$ and 0 to all other agents. Finally, let

$$E^{ij} \equiv \{f \in E : f = f_i^{|i} + f_j^{|j}\} \qquad E_i^{ij} \equiv \{f_i \in F : f \in E^{ij}\}$$

which denotes efficient allocations that only allocates resources to agents $i$ and $j$.

## 7.1 Concerned Lesser Evil

A first extension retains the notion that consensus involves only agents which are concerned in a pairwise comparison of two evil allocations. In other words, to $f_i^{|i}$ and $g_j^{|j}$, provided that $(f_i, g_j) \in E_i \times E_j$, the consensus of the concerned agents only (that is, $i$ and $j$) is sufficient.

Concerned Lesser Evil: For all $i, j \in N$, $f_i, g_j \in E_i^{ij}$, if $f_i \succeq_i g_j$ and $f_i \succeq_j g_j$, then $f_i^{|i} \succeq g_j^{|j}$, the relation being strict if $f_i \succ_i g_j$ or $f_i \succ_j g_j$.

We propose a representation to generalize the mixed benchmark representation of Section 2, in which Definition 2 is modified as follows.

**Definition 8.** *A* $(m_i^j)_{\substack{i \in N \\ j \in N}}$ *where* $m_i^j$ *is a score metric representing* $\succeq_u$ *is a* benchmark system *if for all* $j \in N$ *and all* $f_i \in E_i^j$,

$$m_i^i(f_i) = \sum_{k \in N \setminus \{j\}} m_j^k(f_i).$$

Note that if $|N| = 2$ then the benchmark system

$$\begin{pmatrix} m_1^1 & m_1^2 \\ m_2^1 & m_2^2 \end{pmatrix}$$

is such that for all $i = 1, 2$ and $f_i \in E_i$, $m_i^i(f_i) = m_{-i}^i(f_i)$. Benchmark systems therefore offer a genuine generalization of benchmark metrics. One can therefore mix them as proposed in Definition 9.

**Definition 9.** *A metric* $\mathbf{m} = (m_1, \ldots, m_n)$ mixes benchmark systems *if there exists a benchmark system* $(m_i^j)_{\substack{i \in N \\ j \in N}}$ *such that* $\mathbf{m} = \frac{\sum_{j \in N} \mathbf{m}^j}{n}$.

A social preference represented by a metric that mixes benchmark systems satisfies Concerned Lesser Evil. Indeed, if $f_i^{|i} \sim f_j^{|j}$, then

$$\begin{aligned} & m_i^i(f_i) + \sum_{k \in N \setminus \{i\}} m_i^k(f_i) = m_j^j(g_j) + \sum_{k \in N \setminus \{j\}} m_j^k(g_j) \\ \implies & \sum_{k \in N \setminus \{i\}} m_i^k(f_i) - m_j^j(g_j) = \sum_{k \in N \setminus \{j\}} m_j^k(g_j) - m_i^i(f_i). \end{aligned}$$

Hence, by Definition 8 this is equivalent to

$$\sum_{k \in N \setminus \{i\}} m_i^k(f_i) - \sum_{k \in N \setminus \{i\}} m_i^k(g_j) = \sum_{k \in N \setminus \{j\}} m_j^k(g_j) - \sum_{k \in N \setminus \{j\}} m_j^k(f_i)$$

which yields $f_i \succeq_i g_j$ if and only if $g_j \succeq_j f_i$. This confirms that Concerned Lesser Evil is necessary to show that a social preference is represented by a metric that mixes benchmarks. We conjecture the sufficient part of the theorem also holds.

**Theorem 4** (Conjecture)**.** *A pseudo-egalitarian social preference satisfies Concerned Lesser Evil if and only if it is represented by a metric* $\mathbf{m}$ *that mixes benchmark systems.*

### 7.2 Global Lesser Evil

A second extension requires a stronger notion of consensus, which must now involve all agents. In other words, to compare $f_i^{|i}$ and $g_j^{|j}$, provided that $(f_i, g_j) \in E_i^{ij} \times E_j^{ij}$, the consensus of all agents is required. This means that agents who are not directly concerned should have a say in how $f_i^{|i}$ and $g_j^{|j}$ may be ranked.

To define such axiom we need to sp

Global Lesser Evil: For all $i, j \in N$, $f_i, g_j \in E_i^{ij}$, if for all $k \in N$, $f_i \succeq_k g_j$, then $f_i^{|i} \succeq g_j^{|j}$, the relation being strict if $f_i \succ_k g_j$ for some $k \in N$.

We leave the identification of the representation for such axiom for further works.

## 8 General Spine-Benchmark Interpretation

Consider an allocation that allocates the worst possible act to one agent and something to the other agent, that is $(f_1, 0)$ or $(0, g_2)$. Intuitively both are unfair and can be coined as "evil". Our criterion states however that a lesser evil can be identified if both agent agree that agent is better with $f_1$, then agent is with $g_2$.

Ubiquitous Consensus Lesser Evil (UCLE): for all $f_1, g_2 \in F$, if $f_1 \succsim_1 g_2$ and $f_1 \precsim_2 g_2$, then $(f_1, 0) \succsim (0, g_2)$, with strict in case one agent has a strict preference.

While this formulation makes sense, it clashes with Pareto optimality. To see why, consider $f_1, g_2 \in F$ such that $(f_1, 0) \sim (0, g_2)$ and $f_1 \succ_1 g_2$ and $g_2 \succ_2 f_1$—which is one possibility left by the axiom. By continuity, this implies that there exists $f_1' \sim_1 f_1 \succ_1 g_2$ and $f_1' \succ_2 g_2$—see Figure 3. By Pareto, $f_1' \sim_1 f_1$ entails $(f_1', 0) \sim (f_1, 0) \sim (0, g_2)$, which is impossible by UCLE. The argument is summarized by the following proposition.

**Proposition 11.** *If* $\succsim_1 \neq \succsim_1$*, then there exists no social preference satsfying both Pareto and UCLE.*

This impossibility suggests refining UCLE. Such a requirement can we elaborated with the notion of *spine*. A spine consists in a set of act that comprehensively and uniquely intersects with the indifference of all agents.

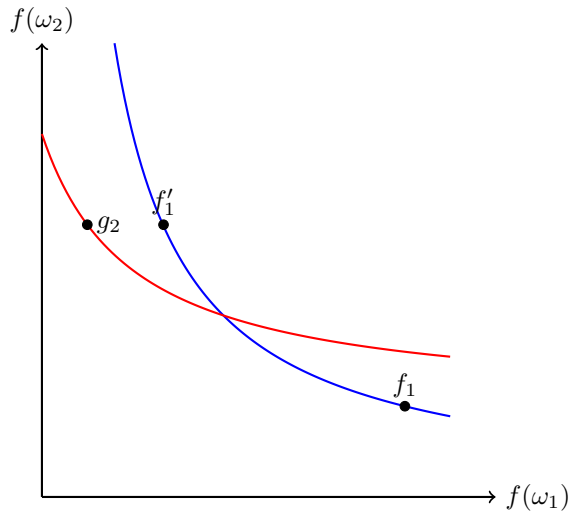



Figure 3: UCLE clashes Pareto

**Definition 10.** *The set $S \subseteq F$ is* a spine *if for all $i \in \{1,2\}$ and all $f_i$, there exists a unique $g_i \in S$ such that $f_i \sim_i g_i$.*

Our main axiom can now be refined by imposing that it hold true only for a given spine.

$S_1 \times S_2$-Consensus Lesser Evil $S_1 \times S_2$-CLE: for all $(f_1, g_2) \in S_1 \times S_2$, if $f_1 \succsim_1 g_2$ and $f_1 \succsim_2 g_2$, then $(f_1, 0) \succsim (0, g_2)$, with strict in case one agent has a strict preference.

The following proposition confirms that restricting $S_1 \times S_2$-CLE to hold on a spine makes it compatible with Pareto.

**Proposition 12.** *There exists a social preference satisfying $S_1 \times S_2$-CLE and Pareto.*

As the representation makes clear, one should interpret the spine as a common ground on which the agents' welfare can quantified. To see what we obtain with this axiom, consider a spine pair $S_1 \times S_2$ and define a metric based on a spine $S_1 \times S_2$ as follows:

**Definition 11.** *Consider a spine $S_1 \times S_2$, a metric $\mathbf{m}^i = (m_1^i, m_2^i)$ is* spined by $i$ *if for all $f_i \in S_i$, $m_i^i(f_i) = m_i^{-i}(f_i)$. A metric mixes spines if $\mathbf{m} = \mathbf{m}^1 + \mathbf{m}^2$.*

From $S_1 \times S_2$-CLE we obtain a Theorem similar to Theorem 1, expept that the spine not based on the Pareto set.

**Theorem 5.** *A pseudo-egalitarian social preference satisfies $S_1 \times S_2$-CLE if and only if it is represented by a metric $\mathbf{m}$ that mixes spines.*

The concept of spine allows for recovering the possible welfare measures in the literature. Among them one can distinguish objective spines, where both agents have the same prospect over the spine $S_1 = S_2$, and subjective spines, where their perspectives differ $S_1 \neq S_2$.

An example of objective spine is the certainty equivalent, which requires that both score metrics equalize on every allocations that yield the same consequence in every states: for all $f_i \in S_i$ where

$$S_i = \{f_i \in F : f_i(\omega) = f_i(\omega') \forall \omega, \omega' \in \Omega\},$$

$m_i^i(f_i) = m_j^i(f_i)$.

In contrast, the subjective spine of Theorem 1 is based on a subjective spine.

# 9 Example with Cobb-Douglas Preferences

There are two representatives, West and East, denoted by $w$ and $e$. Each feasible arrangement assigns a budget share and a share of administrative control. This fits directly into the two-dimensional act framework of Section 1. For transparency, suppose West and East rank bundles by

$$\nu_w(f^b, f^p) = (f^b)^{2/3}(f^p)^{1/3}, \qquad \nu_e(f^b, f^p) = (f^b)^{1/3}(f^p)^{2/3}. \tag{5}$$

West therefore places relatively more weight on budget, while East places relatively more weight on control. The Cobb–Douglas form is only a convenient way to encode that disagreement. What matters is the multidimensional conflict it captures. The same institutional compromise can look harsher or milder depending on which dimension receives more weight, which is exactly the kind of setting in which local comparability rather than a global interpersonal scale becomes decisive (Fleurbaey and Maniquet, 2011; Fleurbaey and Schokkaert, 2013).

Pareto efficiency reduces the problem to a one-parameter frontier. Appendix H shows that every interior efficient allocation can be written as

$$\mathbf{f}(p) = \big((b(p), p), (1 - b(p), 1 - p)\big), \qquad b(p) = \frac{4p}{1 + 3p}, \qquad p \in (0, 1). \tag{6}$$

As $p$ rises, the compromise shifts toward West in both dimensions. The frontier is therefore a one-dimensional family of opposed efficient allocations of the kind studied throughout the paper. Theorem 3 fixes the ordinal leximin form on that frontier, but it does not determine how East's disadvantaged positions should be normalized against West's. That is the remaining comparison problem in the example.

Envy-freeness treats the frontier point by point. Along the frontier, it keeps only allocations at which neither side would rather swap positions. Appendix H shows that the envy-free part of the frontier is exactly

$$\mathbf{f}(p) \in EF \quad \Longleftrightarrow \quad p \in [p^w, p^e] \approx [0.284, 0.386]. \tag{7}$$

This corridor is informative but limited. It ranks efficient points inside the corridor above efficient points outside it, but it does not tell the planner how to compare two distinct efficient points inside the corridor or two points outside it. In institutional environments such as reunification, feasible packages may also be coarse, so the feasible set may miss the corridor altogether. Swap-based fairness then remains normatively relevant, but it does not settle the remaining efficient choice.

CLE asks a different question. It does not ask whether a single efficient point survives a swap test. It asks whether, when two efficient compromises favor different sides, both parties agree on which disadvantaged position is the lesser evil. Compare first $\mathbf{f}(0.55)$ and $\mathbf{f}(0.45)$. Under $\mathbf{f}(0.55)$, West receives approximately $(0.830, 0.55)$ and East $(0.170, 0.45)$. Under $\mathbf{f}(0.45)$, West receives approximately $(0.766, 0.45)$ and East $(0.234, 0.55)$. The relevant disadvantaged bundles are therefore East's bundle under $\mathbf{f}(0.55)$ and West's bundle under $\mathbf{f}(0.45)$. These bundles have the same control level, but the latter has more budget. Since both parties rank bundles monotonically in both dimensions, both regard West's disadvantaged bundle at $\mathbf{f}(0.45)$ as better. The lesser-evil comparison is therefore unanimous, and CLE rules out indifference and favors $\mathbf{f}(0.45)$.

The contrast appears when that local agreement disappears. Compare $\mathbf{f}(0.55)$ with $\mathbf{f}(0.20)$. Under $\mathbf{f}(0.20)$, West receives $(0.500, 0.20)$ and East $(0.500, 0.80)$. The disadvantaged positions are now East's bundle $(0.170, 0.45)$ under $\mathbf{f}(0.55)$ and West's bundle $(0.500, 0.20)$ under $\mathbf{f}(0.20)$. The former has less budget but more control, while the latter has more budget but less control. West, who places relatively more weight on budget, prefers $(0.500, 0.20)$. East, who places relatively more weight on control, prefers $(0.170, 0.45)$. There is therefore no common lesser-evil judgment, and CLE is silent. That silence is substantive rather than a defect. In a genuinely multidimensional dispute, some efficient compromises are publicly

orderable because both sides recognize the same lesser evil, while others remain unresolved because the burden itself is contested.

# 10 Conclusion

This paper studies how far fair social rankings can be justified when individuals have heterogeneous preferences and the admissible informational basis is purely ordinal. Once Pareto efficiency restricts attention to the efficient set, the central issue is not whether social evaluation should give priority to disadvantage. The central issue is how much of that priority can be justified without assuming a comprehensive interpersonal welfare scale. Theorem 3 answers the first part of that question. Pseudo-egalitarianism recovers an ordinal Rawlsian priority structure and yields an aligned leximin representation. The same theorem also identifies the precise freedom left open by ordinal information alone: the common normalization of the individual indices remains undetermined. In that sense, the paper takes seriously the informational-basis concerns emphasized by Robbins (1935) and Sen (1977a).

The rest of the paper studies two different refinements of that remaining freedom. CLE asks which socially indifferent efficient tradeoffs can survive once the parties themselves agree about which disadvantaged position is the lesser evil. Theorem 1 shows that, within the pseudo-egalitarian class, this local agreement discipline yields a balanced Pareto-spined representation. Theorem **??** shows that the same logic extends to many-agent environments through pairwise local spines rather than a single global scale. Envy-freeness works differently. It has force through the no-envy boundary on the efficient set and thereby anchors the same ordinal leximin structure through within-person swap comparisons, but ultimately has little bite in our framework. The comparison results in Section **??** show that CLE and envy-freeness are therefore distinct refinements of the same normalization problem rather than competing priority principles.

What the paper leaves open is part of its contribution. The results identify which rankings of efficient compromises can be justified without moving beyond the paper's ordinal and public-facing discipline, in the spirit of the justificatory constraints emphasized by Rawls (1993). They do not determine how every remaining dilemma should be completed once agreement about disadvantage runs out or once several publicly defensible comparison bases point in different directions. That residual openness is not a technical gap. It reflects the fact that the choice of a comparison basis is itself a substantive normative commitment, just as the choice of inequality aversion is substantive once one moves from a partial order to a complete social ranking (Atkinson, 1970).

The practical implication is limited but precise. Applied welfare analysis can often say more than unrestricted ordinalism suggests, because legal principles, institutional benchmarks, and the parties' own shared judgments may rule out some efficient indifferences. At the same time, it should not present a uniquely correct completion when those sources of agreement are absent. The relevant applied question is which indicators can credibly sustain agreement about disadvantage on the efficient tradeoffs that matter, and when the remaining incompleteness is the appropriate output of pluralistic evaluation rather than something to be removed by convention (Fleurbaey and Maniquet, 2011; Sen, 1977a).

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# A Technical preliminaries

This appendix records the technical facts used repeatedly in later proofs. It fixes the topology on acts and allocations, states the convergence notion used for sequences of social preferences, isolates the feasibility and frontier facts invoked elsewhere, and collects two one-sided comparison lemmas that recur in the representation arguments. Throughout, let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $X = [0, 1]$. An *act* is a measurable map $f : \Omega \to X$, identified up to $\mu$-almost-everywhere equality. We write $F$ for the resulting quotient space. Statements such as $f \geq g$ and $f_1 + f_2 \leq 1$ are understood pointwise for representatives and are required to hold $\mu$-almost everywhere.

On $F$ we use

$$d_m(f, g) \equiv \int_\Omega \frac{|f(\omega) - g(\omega)|}{1 + |f(\omega) - g(\omega)|}\, d\mu(\omega),$$

and on $F^2$ the product metric

$$d_m^2(f, g) \equiv \max\{d_m(f_1, g_1), d_m(f_2, g_2)\}.$$

Continuity of individual preferences is with respect to $d_m$, and continuity of social preferences on subsets of $F^2$ is with respect to $d_m^2$.

## A.1 Convergence on acts and convergence of social preferences

We first record the topology on acts and the convergence notion for binary relations used in closure arguments. We then fix the metric used on normalized leximin classes.

### A.1.1 Acts

The metric $d_m$ is only a convenient way to write convergence in measure on the bounded act space $F$.

**Lemma 1** (Metricization of convergence in measure)**.** *For $(f^n)_n \subseteq F$ and $f \in F$, one has $d_m(f^n, f) \to 0$ if and only if $f^n \to f$ in $\mu$-measure.*

*Proof.* If $d_m(f^n, f) \to 0$, then for every $\varepsilon > 0$,

$$d_m(f^n, f) \geq \frac{\varepsilon}{1 + \varepsilon}\, \mu\big(\{|f^n - f| > \varepsilon\}\big),$$

so $\mu(\{|f^n - f| > \varepsilon\}) \to 0$.

Conversely, suppose $f^n \to f$ in measure. Since $0 \leq |f^n - f| \leq 1$, for every $\eta \in (0, 1]$,

$$d_m(f^n, f) \leq \eta\, \mu(\Omega) + \mu\big(\{|f^n - f| > \eta\}\big).$$

Taking $\limsup_n$ and then letting $\eta \downarrow 0$ gives $d_m(f^n, f) \to 0$. □

**Lemma 2** (Subsequence principle)**.** *If $f^n \to f$ in $\mu$-measure, then some subsequence converges to $f$ $\mu$-almost everywhere.*

*Proof.* Choose inductively $n_k$ so that

$$\mu\big(\{|f^{n_k} - f| > 1/k\}\big) \leq 2^{-k}.$$

With $A_k := \{|f^{n_k} - f| > 1/k\}$, one has $\sum_k \mu(A_k) < \infty$. Borel–Cantelli therefore gives $\mu(\limsup_k A_k) = 0$, which is exactly $f^{n_k} \to f$ almost everywhere. □

**Lemma 3** (Almost-everywhere convergence implies $d_m$ convergence)**.** *If some representatives satisfy $f^n(\omega) \to f(\omega)$ for $\mu$-almost every $\omega$, then $d_m(f^n, f) \to 0$.*

*Proof.* On a finite measure space, almost-everywhere convergence implies convergence in measure. Apply Lemma 1. □

### A.1.2 Binary relations

For closure arguments we use pointwise stabilization of comparisons. For a relation $\succeq$ on a set $Y$, write $\mathbf{1}_{\succeq}(y, z) = 1$ if $y \succeq z$ and 0 otherwise. A sequence $(\succeq^n)_n$ converges to $\succeq$ if, for every $(y, z) \in Y \times Y$, the indicators $\mathbf{1}_{\succeq^n}(y, z)$ are eventually constant and equal to $\mathbf{1}_{\succeq}(y, z)$.

**Lemma 4** (Finite-test characterization)**.** *For binary relations on a set $Y$, pointwise stabilization is equivalent to eventual stabilization on every finite subset of $Y \times Y$.*

*Proof.* The forward implication follows by taking the maximum of finitely many stabilization times. The converse is the singleton case. □

**Lemma 5** (Limits preserve weak orders)**.** *Let $(\succeq^n)_n$ be weak orders on a set $Y$ that converge to $\succeq$ in the sense of Lemma 4. Then $\succeq$ is a weak order.*

*Proof.* Fix $y, z \in Y$. Since each $\succeq^n$ is complete, for every $n$ at least one of $y \succeq^n z$ or $z \succeq^n y$ holds. Eventual stabilization rules out the possibility that both comparisons are eventually false, so $\succeq$ is complete.

For transitivity, if $x \succeq y$ and $y \succeq z$, then for all large $n$ one has $x \succeq^n y$ and $y \succeq^n z$. Transitivity of $\succeq^n$ gives $x \succeq^n z$ for all large $n$, hence $x \succeq z$. □

**Lemma 6** (Closedness of fixed comparison restrictions)**.** *Let $R \subseteq Y \times Y$. If $(\succeq^n)_n$ converges to $\succeq$ as in Lemma 4 and every $\succeq^n$ satisfies $y \succeq^n z$ for all $(y, z) \in R$, then $\succeq$ satisfies the same restrictions.*

*Proof.* Fix $(y, z) \in R$. Since $y \succeq^n z$ for all $n$, eventual stabilization implies $y \succeq z$. □

### A.1.3 Normalized leximin classes

The next definition only fixes notation for convergence within normalized leximin classes. Fix any finite measure $\nu$ on a $\sigma$-field containing the $d_m$-Borel sets on $F$, and define

$$\mathrm{dist}_\nu(U, V) \equiv \int_F \frac{|U(f) - V(f)|}{1 + |U(f) - V(f)|}\, d\nu(f)$$

for measurable $U, V : F \to [0, 1]$. Also fix a continuous representation $u_1 : F \to [0, 1]$ of $\succeq_1$ with $u_1(0) = 0$ and $u_1(1) = 1$.

**Definition 12** (Leximin distance on social preferences)**.** *Let $\mathcal{P}^{\mathrm{lex}}$ be the set of social preferences $\succeq$ on $\mathcal{F}$ for which there exists a unique measurable $u_2^{\succeq} : F \to [0, 1]$ with $u_2^{\succeq}(0) = 0$ and $u_2^{\succeq}(1) = 1$ such that, for all $f, g \in \mathcal{F}$,*

$$f \succeq g \quad \Longleftrightarrow \quad \big(u_1(f_1), u_2^{\succeq}(f_2)\big) \geq_{\mathrm{Lex}} \big(u_1(g_1), u_2^{\succeq}(g_2)\big).$$

*For $\succeq, \succeq' \in \mathcal{P}^{\mathrm{lex}}$, let*

$$d^{\mathrm{lex}}(\succeq, \succeq') \equiv \mathrm{dist}_\nu\big(u_2^{\succeq}, u_2^{\succeq'}\big).$$

## A.2 Feasibility and Pareto-frontier geometry

We next record the two facts about feasibility and efficiency used later: feasibility is closed under convergence in $F^2$, and compact feasibility together with continuous utilities guarantees at least one efficient allocation.

The feasible set is

$$\mathcal{F} \equiv \{(f_1, f_2) \in F^2 : f_1 + f_2 \leq 1 \ \mu\text{-almost everywhere}\}.$$

An allocation $f \in \mathcal{F}$ *dominates* $g \in \mathcal{F}$ if $f_i \succeq_i g_i$ for both $i$ and $f_j \succ_j g_j$ for some $j$. The Pareto set is

$$E \equiv \{f \in \mathcal{F} : \text{no } g \in \mathcal{F} \text{ dominates } f\},$$

with coordinate projections

$$E_i \equiv \{f_i \in F : \exists f_{-i} \in F \text{ such that } (f_i, f_{-i}) \in E\}, \qquad i \in \{1, 2\}.$$

**Lemma 7** (Closedness and convexity of feasibility)**.** *$\mathcal{F}$ is convex and $d_m^2$-closed.*

*Proof.* Convexity is immediate. For closedness, let $f^n \in \mathcal{F}$ and $f^n \to f$ in $d_m^2$. By Lemmas 1 and 2, after passing to a subsequence we may assume $f_i^n(\omega) \to f_i(\omega)$ for $i = 1, 2$ and $\mu$-almost every $\omega$. Since $f_1^n + f_2^n \leq 1$ almost everywhere for every $n$, the same inequality holds in the limit almost everywhere. Hence $f \in \mathcal{F}$. □

**Lemma 8** (Existence of efficient allocations)**.** *If $\mathcal{F}$ is compact in $d_m^2$ and each $\succeq_i$ admits a continuous utility representation $u_i : F \to \mathbb{R}$, then $E \neq \varnothing$.*

*Proof.* The function $W(f) := u_1(f_1) + u_2(f_2)$ is continuous on the compact set $\mathcal{F}$, so it attains a maximum at some $\hat{f} \in \mathcal{F}$. If $\hat{f} \notin E$, some $g \in \mathcal{F}$ dominates $\hat{f}$, and then $W(g) > W(\hat{f})$, a contradiction. □

Under Assumption **??**, the paper imposes the required one-dimensional frontier regularity directly. The lemma above is used only when one wants to derive nonemptiness of $E$ from compactness and continuity.

## A.3 Frontier parametrization and the Pareto projector

Later appendices sometimes need more than Assumption **??**: they need each efficient component to identify a unique frontier counterpart on the other side. We therefore isolate the additional injectivity requirement separately.

**Assumption 3** (Injective frontier parametrization)**.** *Assumption **??** holds, and for each $i \in \{1, 2\}$ the coordinate map $x \mapsto f_i^x$ is injective on $[0, 1]$.*

Under Assumption 3, each map $x \mapsto f_i^x$ is a continuous bijection from the compact space $[0, 1]$ onto the metric space $E_i$, hence a homeomorphism. Let $\psi_i : E_i \to [0, 1]$ denote its inverse.

**Lemma 9** (Frontier counterpart maps)**.** *Under Assumption 3, there are unique homeomorphisms*

$$\pi_1 : E_1 \to E_2 \qquad \text{and} \qquad \pi_2 : E_2 \to E_1$$

*such that $(f_1, \pi_1(f_1)) \in E$ for every $f_1 \in E_1$ and $(\pi_2(g_2), g_2) \in E$ for every $g_2 \in E_2$. They are mutual inverses and satisfy*

$$\pi_1(f_1^x) = f_2^x \qquad \text{and} \qquad \pi_2(f_2^x) = f_1^x \qquad \text{for all } x \in [0, 1].$$

*Proof.* Define $\pi_1(f_1^x) := f_2^x$ and $\pi_2(f_2^x) := f_1^x$. Injectivity of the coordinate maps makes these definitions well defined, and surjectivity is immediate from the definition of $E_i$. Since $\pi_1 = (x \mapsto f_2^x) \circ \psi_1$ and $\pi_2 = (x \mapsto f_1^x) \circ \psi_2$, both maps are continuous; the same formulas show that they are inverses. Uniqueness is immediate because every point of $E_i$ appears on the frontier in exactly one pair. □

A standard sufficient route to Assumption 3 starts from compact $\mathcal{F}$, continuous strictly increasing utility representations, and the constrained problem

$$\max_{f \in \mathcal{F}} u_2(f_2) \qquad \text{subject to} \qquad u_1(f_1) \geq t.$$

If the feasible correspondence in $t$ is nonempty, compact-valued, and continuous and the argmax is single-valued, Berge's maximum theorem implies continuity of the optimizer in

$t$ (Berge, 1963). If that optimizer traces the whole efficient frontier and the coordinate projections are injective along it, Assumption 3 follows.

### A.4 Single-recipient comparison tools

The last two lemmas isolate one-dimensional arguments that recur throughout the appendix. They do not use frontier geometry.

**Lemma 10** (Existence of a cross-person indifference point)**.** *Let $\succeq$ be a $d_m^2$-continuous weak order on $\mathcal{F}$ satisfying Axiom 1 and $(1,0) \sim (0,1)$. Suppose in addition that the individual preferences satisfy the first part of Assumption* **??***. Then for every $f \in F$ there exists $g \in F$ such that*

$$(f,0) \sim (0,g).$$

*Proof.* Fix $f \in F$ and define $\phi(\alpha) := (0, \alpha \cdot 1)$ for $\alpha \in [0,1]$. Let

$$A := \{\alpha \in [0,1] : \phi(\alpha) \succeq (f,0)\}.$$

Because $\phi$ is continuous and the upper contour set of $(f,0)$ is closed, $A$ is closed. It is nonempty because $1 \succeq_1 f$ by monotonicity, hence $(1,0) \succeq (f,0)$ by Pareto, and therefore $(0,1) \succeq (f,0)$ by the normalization $(1,0) \sim (0,1)$.

Let $\alpha^* := \min A$. Then $(0, \alpha^* \cdot 1) \succeq (f,0)$. If $\alpha^* > 0$, choose any sequence $\alpha_n \uparrow \alpha^*$ with $\alpha_n < \alpha^*$. By minimality of $\alpha^*$, $\alpha_n \notin A$, so completeness gives $(f,0) \succeq (0, \alpha_n \cdot 1)$ for every $n$. Since $(0, \alpha_n \cdot 1) \to (0, \alpha^* \cdot 1)$ in $d_m^2$ and the lower contour set of $(f,0)$ is closed, $(f,0) \succeq (0, \alpha^* \cdot 1)$.

If $\alpha^* = 0$, then $(0,0) \succeq (f,0)$. If $f \neq 0$, strict monotonicity in Assumption **??** gives $f \succ_1 0$, so $(f,0)$ Pareto-dominates $(0,0)$, contradicting Axiom 1. Hence $f = 0$, and then trivially $(f,0) \sim (0,0)$.

Thus in all cases $(f,0) \sim (0, \alpha^* \cdot 1)$, so $g = \alpha^* \cdot 1$ works. □

**Lemma 11** (Monotonicity across single-recipient indifferences)**.** *Let $\succeq$ be a weak order on $\mathcal{F}$ satisfying Axiom 1. If*

$$(f_1,0) \sim (0,g_2) \qquad \text{and} \qquad (f_1',0) \sim (0,g_2'),$$

*then*

$$f_1 \succ_1 f_1' \implies g_2 \succeq_2 g_2', \qquad g_2 \succ_2 g_2' \implies f_1 \succeq_1 f_1'.$$

*Proof.* Suppose $f_1 \succ_1 f_1'$. Then $(f_1,0)$ Pareto-dominates $(f_1',0)$, so $(f_1,0) \succ (f_1',0)$. Using the two indifferences gives $(0,g_2) \succ (0,g_2')$. If $g_2' \succ_2 g_2$, then $(0,g_2')$ Pareto-dominates $(0,g_2)$, contradicting Axiom 1. Hence $g_2 \succeq_2 g_2'$.

The second implication is symmetric. □

## B Proofs for Section 6

This appendix proves Theorem 3. The frontier parametrization plays no role in this section. The proof first calibrates the two individual orders through one-sided comparisons, then identifies the social ranking when the same individual is weakly disadvantaged in both allocations, and finally handles the remaining cross-side cases. For $h \in F$, write

$$h^{10} \equiv (h,0) \qquad \text{and} \qquad h^{01} \equiv (0,h).$$

### B.1 Single-recipient calibration

We begin by aligning the two individual orders on a common ordinal scale. The key inputs are the existence of one-sided indifference points from Appendix A and the observation that, on a fixed recipient side, the social order simply reproduces that recipient's own order.

**Lemma 12** (Single-recipient comparisons on a fixed side)**.** *For all $f, g \in F$,*

$$f^{10} \succeq g^{10} \quad \Longleftrightarrow \quad f \succeq_1 g, \qquad f^{01} \succeq g^{01} \quad \Longleftrightarrow \quad f \succeq_2 g.$$

*In particular,*

$$f^{10} \sim g^{10} \Longleftrightarrow f \sim_1 g, \qquad f^{01} \sim g^{01} \Longleftrightarrow f \sim_2 g.$$

*Proof.* We prove the first equivalence. The second is symmetric.

If $f \succ_1 g$, then $f^{10}$ Pareto-dominates $g^{10}$, so Axiom 1 gives $f^{10} \succ g^{10}$. If $f \sim_1 g$, either $f = g = 1$ almost everywhere, in which case $f^{10} = g^{10}$, or else the set $\{\omega : f(\omega) < 1\}$ has positive measure. In the latter case let

$$f^n(\omega) \equiv \min\Big\{f(\omega) + \frac{1}{n}, 1\Big\}.$$

Then $f^n \to f$ in $d_m$ by Lemma 1, and $f^n \succ_1 f \sim_1 g$ for all large $n$. Hence $(f^n, 0) \succ g^{10}$ by Pareto. Since $(f^n, 0) \to f^{10}$ in $d_m^2$ and the upper contour set of $g^{10}$ is closed, continuity of $\succeq$ yields $f^{10} \succeq g^{10}$. By symmetry, also $g^{10} \succeq f^{10}$, so $f^{10} \sim g^{10}$.

Conversely, if $f^{10} \succeq g^{10}$ but not $f \succeq_1 g$, then $g \succ_1 f$, so $g^{10}$ Pareto-dominates $f^{10}$. Axiom 1 gives $g^{10} \succ f^{10}$, a contradiction. □

Fix continuous strictly increasing representations $u_1, u_2$ of $(\succeq_1, \succeq_2)$ from Assumption **??**. By Lemma 10, for each $f \in F$ choose an act $g(f) \in F$ such that

$$f^{10} \sim g(f)^{01}. \tag{8}$$

Define $\alpha$ on the interval $u_1(F)$ by

$$\alpha\big(u_1(f)\big) \equiv u_2(g(f)),$$

and set

$$m_1(f) \equiv \alpha\big(u_1(f)\big), \qquad m_2(f) \equiv u_2(f). \tag{9}$$

**Lemma 13** (Calibration of the individual indices)**.** *The map $\alpha$ is well defined and strictly increasing on $u_1(F)$. Consequently, each $m_i$ is continuous and, for all $f, f' \in F$ and $i \in \{1, 2\}$,*

$$f \succeq_i f' \quad \Longleftrightarrow \quad m_i(f) \geq m_i(f').$$

*Proof.* If $u_1(f) = u_1(f')$, then $f \sim_1 f'$, hence $f^{10} \sim f'^{10}$ by Lemma 12. Combining this with (8) for $f$ and $f'$ gives $g(f)^{01} \sim g(f')^{01}$, so $g(f) \sim_2 g(f')$ by Lemma 12. Therefore $u_2(g(f)) = u_2(g(f'))$, and $\alpha$ is well defined.

If $u_1(f) > u_1(f')$, then $f \succ_1 f'$, so $f^{10} \succ f'^{10}$ by Lemma 12. Using (8) and transitivity, we get $g(f)^{01} \succ g(f')^{01}$, hence $g(f) \succ_2 g(f')$ by Lemma 12. Thus $\alpha(u_1(f)) > \alpha(u_1(f'))$, so $\alpha$ is strictly increasing.

Because $F$ is convex, it is path connected, and the continuous image $u_1(F)$ is therefore an interval. A strictly increasing function on an interval is continuous, so $m_1 = \alpha \circ u_1$ is continuous. Since $m_1$ is a strictly increasing transform of $u_1$, it represents $\succeq_1$. By definition, $m_2 = u_2$, so $m_2$ is continuous and represents $\succeq_2$. □

The calibration matters because it turns one-sided social comparisons into direct comparisons of the two aligned indices.

**Lemma 14** (Single-recipient comparisons and index comparisons)**.** *For all $f, g \in F$,*

$$f^{10} \succeq g^{01} \quad \Longleftrightarrow \quad m_1(f) \geq m_2(g).$$

*Consequently,*

$$f^{10} \sim g^{01} \quad \Longleftrightarrow \quad m_1(f) = m_2(g).$$

*Proof.* By (8), $f^{10} \sim g(f)^{01}$. Hence

$$f^{10} \succeq g^{01} \quad \Longleftrightarrow \quad g(f)^{01} \succeq g^{01} \quad \Longleftrightarrow \quad g(f) \succeq_2 g,$$

where the second equivalence is Lemma 12. Since $m_2 = u_2$ represents $\succeq_2$, this is equivalent to $u_2(g(f)) \geq u_2(g) = m_2(g)$. By construction, $u_2(g(f)) = \alpha(u_1(f)) = m_1(f)$, so the first claim follows. The indifference statement is the weak comparison in both directions. □

**Lemma 15** (Endpoint alignment of the calibrated indices)**.** *The calibrated indices satisfy*

$$m_1(0) = m_2(0) \qquad \textit{and} \qquad m_1(1) = m_2(1).$$

*Proof.* Since $0^{10} = 0^{01}$, Lemma 14 with $f = g = 0$ gives $m_1(0) = m_2(0)$. Axiom 2 gives $1^{10} \sim 1^{01}$, so Lemma 14 with $f = g = 1$ gives $m_1(1) = m_2(1)$. □

## B.2 Recovering the leximin order on a common disadvantaged side

We now identify the social ranking on the domain where the same individual is weakly disadvantaged in both allocations. On that domain, Axiom 4 fixes the comparison of minima and Axioms 1 and 3 determine the tie-break.

**Lemma 16** (Leximin on a common disadvantaged side)**.** *Let $f, g \in \mathcal{F}$, and write $\mathbf{a} \equiv \mathbf{m}(f)$ and $\mathbf{b} \equiv \mathbf{m}(g)$. If either*

$$a_1 \leq a_2 \quad \textit{and} \quad b_1 \leq b_2,$$

*or*

$$a_2 \leq a_1 \quad \textit{and} \quad b_2 \leq b_1,$$

*then*

$$f \succeq g \quad \Longleftrightarrow \quad \mathbf{a} \geq_{\text{Lex}} \mathbf{b}.$$

*Proof.* By symmetry it is enough to treat the case $a_1 \leq a_2$ and $b_1 \leq b_2$, so agent 1 is weakly disadvantaged in both allocations.

If $a_1 > b_1$, then $\underline{\mathbf{a}} > \underline{\mathbf{b}}$. If $b_1 < b_2$, agent 1 is strictly disadvantaged in $g$, so Axiom 4 applied to $(g, f)$ yields

$$g \succeq f \quad \Longleftrightarrow \quad g_1 \succeq_1 f_1.$$

Since $m_1$ represents $\succeq_1$ and $a_1 > b_1$, the right-hand side is false, so $f \succ g$. If instead $b_1 = b_2$, then $a_1 > b_1$ and $a_2 \geq a_1 > b_2$, so $f$ Pareto-dominates $g$ and again $f \succ g$.

Now suppose $a_1 = b_1$. If $a_2 > b_2$, then $f_1 \sim_1 g_1$ and $f_2 \succ_2 g_2$, so $f$ Pareto-dominates $g$ and $f \succ g$. If $a_2 = b_2$ and $a_1 < a_2$, then agent 1 is strictly disadvantaged in both allocations, and Axiom 4 gives

$$f \succeq g \quad \Longleftrightarrow \quad f_1 \succeq_1 g_1.$$

Because $a_1 = b_1$, the right-hand side holds, and the symmetric argument gives $g \succeq f$, so $f \sim g$. If $a_2 = b_2$ and $a_1 = a_2$, then both allocations are balanced. Lemma 14 gives

$$f_1^{10} \sim g_2^{01} \qquad \text{and} \qquad g_1^{10} \sim f_2^{01},$$

so Axiom 3 yields $f \sim g$.

These cases are exactly the leximin comparison when agent 1 is weakly minimal in both allocations. The case where agent 2 is weakly minimal in both allocations is symmetric. □

## B.3 The cross-side cases

It remains to treat the cases in which different individuals are weakly disadvantaged in the two allocations. The argument is to contract one coordinate inside the relevant allocation until the minima line up. Since the contraction is internal, feasibility is preserved automatically.

**Lemma 17** (Contractions hit every intermediate calibrated value)**.** *Fix $i \in \{1,2\}$ and $h \in F$. If $t \in [m_i(0), m_i(h)]$, then some $\lambda \in [0,1]$ satisfies*

$$m_i(\lambda h) = t.$$

*Proof.* The map $\lambda \mapsto m_i(\lambda h)$ is continuous on $[0,1]$, with values $m_i(0)$ at $\lambda = 0$ and $m_i(h)$ at $\lambda = 1$. The intermediate value theorem gives the claim. □

**Lemma 18** (pseudo-egalitarianism implies the leximin representation)**.** *Let $\succeq$ be pseudo-egalitarian, and let $\mathbf{m} = (m_1, m_2)$ be the calibrated pair above. Then, for all $f, g \in \mathcal{F}$,*

$$f \succeq g \quad \Longleftrightarrow \quad \mathbf{m}(f) \geq_{\text{Lex}} \mathbf{m}(g).$$

*Proof.* Fix $f, g \in \mathcal{F}$, and write $\mathbf{a} \equiv \mathbf{m}(f)$ and $\mathbf{b} \equiv \mathbf{m}(g)$. If the same individual is weakly disadvantaged in both allocations, the claim is Lemma 16. By symmetry it remains only to consider the cross-side case

$$a_1 \leq a_2 \qquad \text{and} \qquad b_2 \leq b_1.$$

If $a_1 > b_2$, then $\underline{\mathbf{a}} > \underline{\mathbf{b}}$. Since $m_1(0) = m_2(0) \leq a_1 \leq a_2 = m_2(f_2)$, Lemma 17 yields $\lambda \in [0,1]$ with $m_2(\lambda f_2) = a_1$. Let $\hat{f} \equiv (f_1, \lambda f_2)$. Then $\hat{f} \in \mathcal{F}$ and $\mathbf{m}(\hat{f}) = (a_1, a_1)$. Lemma 16 applied first to $(f, \hat{f})$ and then to $(\hat{f}, g)$ gives $f \succeq \hat{f} \succ g$, hence $f \succ g$.

If $a_1 < b_2$, the symmetric argument contracts $g_1$ to some $\hat{g} = (\lambda g_1, g_2) \in \mathcal{F}$ with $\mathbf{m}(\hat{g}) = (b_2, b_2)$, and Lemma 16 gives $g \succeq \hat{g} \succ f$. Hence $g \succ f$.

It remains to treat $a_1 = b_2$, so the minima coincide. If also $a_2 = b_1$, then

$$m_1(f_1) = m_2(g_2) \qquad \text{and} \qquad m_1(g_1) = m_2(f_2),$$

so Lemma 14 gives $f_1^{10} \sim g_2^{01}$ and $g_1^{10} \sim f_2^{01}$. Axiom 3 therefore yields $f \sim g$.

If instead $a_2 > b_1$, then $m_2(0) \leq b_1 < a_2 = m_2(f_2)$, so Lemma 17 yields $\lambda \in [0,1]$ with $m_2(\lambda f_2) = b_1$. Let $\tilde{f} \equiv (f_1, \lambda f_2)$. Then

$$m_1(\tilde{f}_1) = a_1 = b_2 = m_2(g_2) \qquad \text{and} \qquad m_2(\tilde{f}_2) = b_1 = m_1(g_1).$$

Lemma 14 therefore gives $\tilde{f}_1^{10} \sim g_2^{01}$ and $g_1^{10} \sim \tilde{f}_2^{01}$, so Axiom 3 implies $\tilde{f} \sim g$. Since agent 1 is weakly disadvantaged in both $f$ and $\tilde{f}$, their minima coincide at $a_1$, and $a_2 > b_1 = m_2(\tilde{f}_2)$, Lemma 16 gives $f \succ \tilde{f}$. Hence $f \succ g$.

The case $a_2 < b_1$ is symmetric and gives $g \succ f$. These cases are exactly the leximin comparison in the remaining cross-side domain. □

## B.4 Proof of Theorem 3

We now assemble the preceding lemmas and then turn to the converse and identification statements.

*Proof of Theorem 3.* Assume first that $\succeq$ is pseudo-egalitarian. Lemma 13 constructs continuous indices $\mathbf{m} = (m_1, m_2)$ that represent $(\succeq_1, \succeq_2)$. Lemma 18 shows that $\succeq$ is represented by leximin applied to $\mathbf{m}$, and Lemma 15 gives

$$m_1(0) = m_2(0) \qquad \text{and} \qquad m_1(1) = m_2(1).$$

Thus $\mathbf{m}$ is an aligned leximin representation.

Conversely, suppose $\mathbf{m} = (m_1, m_2)$ is an aligned leximin representation of $\succeq$. Pareto is immediate: strict Pareto dominance strictly increases the sorted pair and therefore the leximin value. Symmetry is also immediate, because

$$\mathbf{m}(1,0) = \big(m_1(1), m_2(0)\big) \qquad \text{and} \qquad \mathbf{m}(0,1) = \big(m_1(0), m_2(1)\big)$$

have the same minimum and the same maximum by endpoint alignment.

For Axiom 4, fix $f, g \in \mathcal{F}$ and $i \in \{1, 2\}$ with

$$f^{i0} \succ f^{-i0} \qquad \text{and} \qquad g^{i0} \succeq g^{-i0}.$$

Because both one-sided allocations have the common parked level $m_1(0) = m_2(0)$, these premises are equivalent to

$$m_i(f_i) > m_{-i}(f_{-i}) \qquad \text{and} \qquad m_i(g_i) \geq m_{-i}(g_{-i}).$$

Hence coordinate $-i$ is weakly minimal in both allocations and strictly minimal in $f$. Leximin therefore compares $f$ and $g$ through that coordinate:

$$f \succeq g \iff m_{-i}(f_{-i}) \geq m_{-i}(g_{-i}) \iff f_{-i} \succeq_{-i} g_{-i}.$$

For Axiom 3, suppose

$$f^{i0} \sim g^{-i0} \qquad \text{and} \qquad g^{i0} \sim f^{-i0}.$$

Again using the common parked level, these conditions are equivalent to

$$m_i(f_i) = m_{-i}(g_{-i}) \qquad \text{and} \qquad m_i(g_i) = m_{-i}(f_{-i}).$$

Thus $\mathbf{m}(f)$ and $\mathbf{m}(g)$ have the same minimum and the same maximum, so leximin yields $f \sim g$. Therefore $\succeq$ is pseudo-egalitarian.

Finally, consider identification. If $m_i' = \phi \circ m_i$ for both $i$ and some strictly increasing homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, then $\mathbf{m}'$ is again aligned and induces the same leximin ranking, so it is another aligned leximin representation.

For necessity, let $\mathbf{m} = (m_1, m_2)$ and $\mathbf{m}' = (m_1', m_2')$ be aligned leximin representations of the same social preference. Define $\phi : m_1(F) \to m_1'(F)$ by

$$\phi\big(m_1(f)\big) \equiv m_1'(f).$$

This is well defined because $m_1(f) = m_1(f')$ implies $f \sim_1 f'$, hence $m_1'(f) = m_1'(f')$. It is strictly increasing because both $m_1$ and $m_1'$ represent $\succeq_1$. Now fix $g \in F$. By Lemma 10, choose $f \in F$ with $f^{10} \sim g^{01}$. Since both $\mathbf{m}$ and $\mathbf{m}'$ are aligned leximin representations,

$$m_1(f) = m_2(g) \qquad \text{and} \qquad m_1'(f) = m_2'(g).$$

Hence

$$m_2'(g) = m_1'(f) = \phi\big(m_1(f)\big) = \phi\big(m_2(g)\big).$$

Because $g$ was arbitrary, $m_2' = \phi \circ m_2$ on $F$, and by construction $m_1' = \phi \circ m_1$ on $F$ as well.

Since $F$ is convex, it is path connected. Therefore $m_1(F)$ and $m_1'(F)$ are intervals, and $\phi$ is a strictly increasing bijection between intervals. Such a map is a homeomorphism, and any strictly increasing homeomorphism between intervals extends to a strictly increasing homeomorphism of $\mathbb{R}$. This proves the identification statement. □

# C Proofs for Section 2: Theorem 1

This appendix proves the characterization in Section 2. Throughout, individual preferences satisfy Assumption **??**, the efficient frontier satisfies Assumption **??**, and social preferences are continuous weak orders on $\mathcal{F}$. The proof has three parts. We first turn the dilemma boundary into a one-dimensional object indexed by common one-sided indifference levels. We then construct, on each maximal interval of genuine disagreement, a local pair of indices satisfying the balance identity that later defines the collective spine. Finally, we glue those local pairs, extend them to the full domain, and recover the social ranking from single-recipient comparisons.

## C.1 Boundary restriction under CLE

The geometric construction below uses only the negative boundary content of CLE. By Lemma **??**, if $(f_1, g_2) \in \mathcal{D}^{\succeq}$, then neither $f_1$ nor $g_2$ can be unanimously weakly preferred to the other with at least one strict comparison. On the dilemma boundary, unanimous agreement can therefore fail only through genuine disagreement or common indifference.

## C.2 The dilemma boundary and disagreement intervals

We begin by parametrizing the dilemma boundary. Under pseudo-egalitarianism, every efficient position can be indexed by its common single-recipient indifference level, and that index orders both dilemma projections.

**Lemma 19** (Parametrization of the dilemma boundary)**.** *Assume $\succeq$ is pseudo-egalitarian. Then there are homeomorphisms*

$$x \longmapsto d_1^x \in \mathcal{D}_1^{\succeq}, \qquad x \longmapsto d_2^x \in \mathcal{D}_2^{\succeq}, \qquad x \in [0,1],$$

*such that $(d_1^x, 0) \sim (0, d_2^x)$ for every $x$, and, whenever $x < y$,*

$$d_1^y \succ_i d_1^x \qquad \text{and} \qquad d_2^y \succ_i d_2^x \qquad \text{for both } i \in \{1,2\}.$$

*In particular,*

$$\mathcal{D}_1^{\succeq} = E_1 \qquad \text{and} \qquad \mathcal{D}_2^{\succeq} = E_2.$$

*Proof.* By Theorem 3, $\succeq$ admits an aligned leximin representation through continuous indices $\mathbf{m} = (m_1, m_2)$. Post-compose both coordinates with the same strictly increasing transformation so that

$$m_1(0) = m_2(0) = 0, \qquad m_1(1) = m_2(1) = 1.$$

By Assumption **??**, after reparametrizing the efficient frontier we may write

$$x \longmapsto f^x = (f_1^x, f_2^x), \qquad x \in [0,1],$$

with $f^0 = (0,1)$ and $f^1 = (1,0)$, where $x \mapsto f_1^x$ is strictly increasing for both individuals and $x \mapsto f_2^x$ is strictly decreasing for both individuals. Define

$$a(x) := m_1(f_1^x), \qquad c(x) := m_2(f_2^x).$$

Then $a$ is a continuous strictly increasing bijection of $[0,1]$ onto $[0,1]$, and $c$ is a continuous strictly decreasing bijection of $[0,1]$ onto $[0,1]$. Let

$$\tau := c^{-1} \circ a.$$

Thus $\tau$ is a continuous strictly decreasing bijection of $[0,1]$ onto itself. Set

$$d_1^x := f_1^x, \qquad d_2^x := f_2^{\tau(x)}.$$

Then

$$m_1(d_1^x) = a(x) = c(\tau(x)) = m_2(d_2^x),$$

so $(d_1^x, 0) \sim (0, d_2^x)$ by Lemma 14. Monotonicity along each branch is immediate from Assumption **??** and the monotonicity of $\tau$.

It remains to identify the images. Because $d_1^x = f_1^x$, the image of $x \mapsto d_1^x$ is $E_1$. Likewise, because $\tau$ is onto, the image of $x \mapsto d_2^x$ is $E_2$. Since each pair $(d_1^x, d_2^x)$ lies on the dilemma boundary, these images are exactly $\mathcal{D}_1^{\succeq}$ and $\mathcal{D}_2^{\succeq}$. Continuity and compactness imply that both maps are homeomorphisms onto their images. □

The next step is to isolate the intervals on which the two individuals strictly disagree about which boundary branch is better.

**Definition 13** (Disagreement intervals)**.** *Let $\mathcal{I}$ be the family of maximal closed intervals $I = [x_I, x^I] \subseteq [0,1]$ such that*

$$d_1^x \not\sim_i d_2^x \qquad \text{for every } x \in (x_I, x^I) \text{ and both } i \in \{1,2\}.$$

*For $I \in \mathcal{I}$, write*

$$\mathcal{D}_1^I := \{d_1^x : x \in I\}, \qquad \mathcal{D}_2^I := \{d_2^x : x \in I\}.$$

The first structural consequence of CLE is that one agent's indifference on the boundary forces the other's.

**Lemma 20** (Common indifference at interval endpoints)**.** *Assume* CLE*. If $d_1^x \sim_1 d_2^x$ for some $x \in [0,1]$, then $d_1^x \sim_2 d_2^x$. Consequently, for every $I = [x_I, x^I] \in \mathcal{I}$,*

$$d_1^{x_I} \sim_1 d_2^{x_I} \iff d_1^{x_I} \sim_2 d_2^{x_I}, \qquad d_1^{x^I} \sim_1 d_2^{x^I} \iff d_1^{x^I} \sim_2 d_2^{x^I}.$$

*Proof.* Fix $x$ with $d_1^x \sim_1 d_2^x$. Since $(d_1^x, 0) \sim (0, d_2^x)$, Lemma **??** rules out both $d_1^x \succ_2 d_2^x$ and $d_2^x \succ_2 d_1^x$. Hence $d_1^x \sim_2 d_2^x$.

Now let $I = [x_I, x^I] \in \mathcal{I}$. By maximality, neither endpoint belongs to the interior disagreement set, so at each endpoint at least one individual is indifferent between the two branches. The first part then gives indifference for the other individual as well. □

The second consequence is that the sign of disagreement cannot switch inside one maximal interval.

**Lemma 21** (Constant sign on a disagreement interval)**.** *Assume* CLE *and fix $I = [x_I, x^I] \in \mathcal{I}$. Exactly one of the following two alternatives holds for every $x \in (x_I, x^I)$:*

$$d_1^x \succ_1 d_2^x \quad \text{and} \quad d_2^x \succ_2 d_1^x,$$

*or*

$$d_2^x \succ_1 d_1^x \quad \text{and} \quad d_1^x \succ_2 d_2^x.$$

*Proof.* Let

$$B := \{x \in (x_I, x^I) : d_1^x \succ_1 d_2^x\}.$$

Because $x \mapsto (d_1^x, d_2^x)$ is continuous and $\succeq_1$ is continuous, the set $\{x : d_1^x \succeq_1 d_2^x\}$ is closed in $(x_I, x^I)$. Since there is no indifference in the interior of $I$, the same set is open there. Hence $B$ is clopen in the connected interval $(x_I, x^I)$, so either $B = \varnothing$ or $B = (x_I, x^I)$.

Fix any $x \in (x_I, x^I)$. If $d_1^x \succ_1 d_2^x$, Lemma **??** rules out $d_1^x \succeq_2 d_2^x$, so necessarily $d_2^x \succ_2 d_1^x$. The opposite case is symmetric. □

We now fix one disagreement interval and exploit its one-dimensional structure. The next lemma introduces the two counterpart maps that trace where agent 1 is indifferent across the two boundary branches.

**Lemma 22** (Counterpart maps on one interval)**.** *Assume* CLE*. Fix $I = [x_I, x^I] \in \mathcal{I}$ and suppose*

$$d_1^x \succ_1 d_2^x \qquad \text{and} \qquad d_2^x \succ_2 d_1^x \qquad \text{for every } x \in (x_I, x^I).$$

*Then, for every $x \in (x_I, x^I)$, there are unique points*

$$\eta(x) \in (x_I, x), \qquad \theta(x) \in (x, x^I),$$

*such that*

$$d_1^{\eta(x)} \sim_1 d_2^x, \qquad d_2^{\theta(x)} \sim_1 d_1^x. \tag{10}$$

*Moreover, $\eta$ and $\theta$ are continuous and strictly increasing on $(x_I, x^I)$, with*

$$\eta(x) \downarrow x_I \quad \text{as } x \downarrow x_I, \qquad \theta(x) \uparrow x^I \quad \text{as } x \uparrow x^I.$$

*Proof.* Fix $x \in (x_I, x^I)$. By hypothesis, $d_1^x \succ_1 d_2^x$. At the left endpoint, Lemma 20 gives $d_1^{x_I} \sim_1 d_2^{x_I}$, while Lemma 19 gives $d_2^{x_I} \prec_1 d_2^x$. Hence

$$d_1^{x_I} \prec_1 d_2^x \prec_1 d_1^x.$$

Since $z \mapsto d_1^z$ is continuous and strictly increasing in $\succeq_1$, continuity of $\succeq_1$ yields $\eta(x) \in (x_I, x)$ with $d_1^{\eta(x)} \sim_1 d_2^x$. Uniqueness follows from strict monotonicity of the branch. The construction of $\theta(x)$ is symmetric.

If $x < y$, then $d_2^y \succ_1 d_2^x$, so the unique point on the $d_1$-branch that is indifferent to $d_2^y$ lies to the right of the one indifferent to $d_2^x$. Thus $\eta(y) > \eta(x)$. The same argument gives strict monotonicity of $\theta$.

For continuity, let $x_n \to x$ and suppose $\eta(x_n) \to z$ along a subsequence. Since $d_1^{\eta(x_n)} \sim_1 d_2^{x_n}$ for all $n$, continuity of the branch maps and of $\succeq_1$ yields $d_1^z \sim_1 d_2^x$. By uniqueness, $z = \eta(x)$. Thus every convergent subsequence of $\eta(x_n)$ converges to $\eta(x)$, so $\eta(x_n) \to \eta(x)$. The proof for $\theta$ is identical.

Finally, because $\eta(x) < x$ and $\eta$ is increasing, $\bar{x} := \lim_{x \downarrow x_I} \eta(x)$ exists in $[x_I, x^I)$. Passing to the limit in $d_1^{\eta(x)} \sim_1 d_2^x$ gives $d_1^{\bar{x}} \sim_1 d_2^{x_I}$. Lemma 20 also gives $d_1^{x_I} \sim_1 d_2^{x_I}$, so uniqueness implies $\bar{x} = x_I$. Thus $\eta(x) \downarrow x_I$ as $x \downarrow x_I$. The argument for $\theta$ is symmetric. □

Iterating the two counterpart maps produces the two chains along which the local balance construction will be summed.

**Lemma 23** (Counterpart iterates converge to the endpoints)**.** *Assume the hypotheses of Lemma 22. Fix $x^m \in (x_I, x^I)$ and define*

$$x_0^{\downarrow} := x^m, \qquad x_{n+1}^{\downarrow} := \eta(x_n^{\downarrow}),$$

$$x_0^{\uparrow} := x^m, \qquad x_{n+1}^{\uparrow} := \theta(x_n^{\uparrow}).$$

*Then $(x_n^{\downarrow})_n$ is strictly decreasing with limit $x_I$, and $(x_n^{\uparrow})_n$ is strictly increasing with limit $x^I$.*

*Proof.* Because $\eta(x) < x$ on $(x_I, x^I)$, the sequence $(x_n^{\downarrow})$ is strictly decreasing and bounded below by $x_I$, hence converges to some $\underline{x} \in [x_I, x^m)$. Continuity of $\eta$ gives $\underline{x} = \eta(\underline{x})$. If $\underline{x} \in (x_I, x^I)$, then (10) implies $d_1^{\underline{x}} \sim_1 d_2^{\underline{x}}$, contradicting the definition of $\mathcal{I}$. Hence $\underline{x} = x_I$. The proof for $(x_n^{\uparrow})$ is symmetric. □

## C.3 Local balance on one disagreement interval

Fix $I = [x_I, x^I] \in \mathcal{I}$ and suppose

$$d_1^x \succ_1 d_2^x \qquad \text{and} \qquad d_2^x \succ_2 d_1^x \qquad \text{for every } x \in (x_I, x^I).$$

The opposite sign pattern is symmetric after swapping the roles of the two individuals. Let $\mathcal{U}^I$ denote the set of continuous utility representations of $\succeq_2$ on the compact set $\mathcal{D}_1^I \cup \mathcal{D}_2^I$. For $u \in \mathcal{U}^I$, write

$$\gamma^u(x) := u(d_2^x) - u(d_1^x), \qquad x \in I.$$

Under the maintained sign pattern, $\gamma^u(x) > 0$ on $(x_I, x^I)$ for every $u \in \mathcal{U}^I$.

### C.3.1 Normalizing the gap

We first choose a representation of $\succeq_2$ whose branch gap is single-peaked and summably controlled along the two iterate chains.

**Lemma 24** (Normalization with summable iterate gaps)**.** *There exist $w \in \mathcal{U}^I$, a maximizer $x^m \in (x_I, x^I)$ of $\gamma^w$, and $C > 0$ such that, for the iterate chains from Lemma 23,*

$$\gamma^w(x_n^{\downarrow}) = \frac{C}{(n+1)^2}, \qquad \gamma^w(x_n^{\uparrow}) = \frac{C}{(n+1)^2} \qquad (n \geq 0).$$

*Proof.* Start from any $u \in \mathcal{U}^I$ and choose $x^m$ maximizing $\gamma^u$ on the compact interval $I$. Post-compose $u$ with an increasing affine map so that

$$u(\mathcal{D}_1^I \cup \mathcal{D}_2^I) \subseteq [0,1].$$

Choose $C > 0$ with $\sum_{n\geq 0} C/(n+1)^2 < 1/4$, and set $\delta_n := C/(n+1)^2$.

Consider the countable set

$$S := \{u(d_1^{x_n^\downarrow}), u(d_2^{x_n^\downarrow}), u(d_1^{x_n^\uparrow}), u(d_2^{x_n^\uparrow}) : n \geq 0\}.$$

Because $u$ represents a weak order, $S$ inherits the usual order from $\mathbb{R}$. One can therefore define recursively a strictly increasing map $\phi_0 : S \to \mathbb{R}$ that preserves the four values indexed by $x^m$ and satisfies

$$\phi_0(u(d_2^{x_n^\downarrow})) - \phi_0(u(d_1^{x_n^\downarrow})) = \delta_n,$$

$$\phi_0(u(d_2^{x_n^\uparrow})) - \phi_0(u(d_1^{x_n^\uparrow})) = \delta_n \qquad (n \geq 0).$$

The tail bound on $\sum_n \delta_n$ guarantees enough room at each stage to preserve the strict order on $S$ while imposing the prescribed gaps. Extend $\phi_0$ to a continuous strictly increasing map $\phi : [0,1] \to \mathbb{R}$ by affine interpolation on the components of $[0,1] \setminus S$, and set $w := \phi \circ u$. Then $w \in \mathcal{U}^I$, $x^m$ still maximizes $\gamma^w$, and the required identities hold by construction. □

The next step irons the gap on one subinterval without changing values to its right.

**Definition 14** (Local ironing class)**.** *For $w \in \mathcal{U}^I$ and $x \leq y$ in $I$, let $\mathcal{I}(x,y;w)$ be the set of all $u \in \mathcal{U}^I$ such that $u = w$ on $\mathcal{D}_1^{[y,x^I]} \cup \mathcal{D}_2^{[y,x^I]}$ and $\gamma^u$ is weakly increasing on $[x,y]$.*

**Lemma 25** (One-step ironing)**.** *Let $w \in \mathcal{U}^I$ and $x < y$ in $I$. If $\gamma^w(x) \leq \gamma^w(y)$, then $\mathcal{I}(x,y;w)$ is nonempty.*

*Proof.* Write

$$A_f := w(d_1^y), \quad A_g := w(d_2^y), \quad B_f := w(d_1^x), \quad B_g := w(d_2^x).$$

The hypothesis is $B_g - B_f \leq A_g - A_f$. Define

$$\lambda(t) := \frac{w(d_1^t) - B_f}{A_f - B_f} \in [0,1], \qquad \widehat{\gamma}(t) := \gamma^w(x) + \lambda(t)\big(\gamma^w(y) - \gamma^w(x)\big).$$

Then $\widehat{\gamma}$ is continuous, weakly increasing on $[x,y]$, and matches $\gamma^w$ at the endpoints.

Let

$$M := \max\{w(d_2^t) : t \in I\},$$

and define

$$T := \{w(d_1^t) : t \in [x,y]\} \cup \{w(d_2^t) : t \in [x,y]\} \cup [A_g, M].$$

Define $\psi : T \to \mathbb{R}$ by

$$\psi(t) = t \text{ on } [A_g, M], \qquad \psi(w(d_1^t)) = w(d_1^t), \qquad \psi(w(d_2^t)) = w(d_1^t) + \widehat{\gamma}(t).$$

The two maps $t \mapsto w(d_1^t)$ and $t \mapsto w(d_1^t) + \widehat{\gamma}(t)$ are continuous and strictly increasing on $[x,y]$, agree with the original values at $x$ and $y$, and the second stays strictly above the first on $(x,y)$. Hence $\psi$ is well defined and strictly increasing on the compact ordered set $T$. Extend $\psi$ to a continuous strictly increasing map on $\mathbb{R}$ by affine interpolation and set $u := \psi \circ w$. Then $u \in \mathcal{U}^I$, coincides with $w$ on $\mathcal{D}_1^{[y,x^I]} \cup \mathcal{D}_2^{[y,x^I]}$, and satisfies $\gamma^u = \widehat{\gamma}$ on $[x,y]$. Thus $u \in \mathcal{I}(x,y;w)$. □

**Lemma 26** (Single-peaked gap with summable control)**.** *There exist $u_2^I \in \mathcal{U}^I$ and $x^m \in (x_I, x^I)$ such that $\gamma^{u_2^I}$ is weakly increasing on $[x_I, x^m]$, weakly decreasing on $[x^m, x^I]$,*

$$\gamma^{u_2^I}(x) > 0 \text{ for } x \in (x_I, x^I), \qquad \gamma^{u_2^I}(x_I) = \gamma^{u_2^I}(x^I) = 0,$$

*and, for some $C > 0$,*

$$\gamma^{u_2^I}(x_n^{\downarrow}) = \frac{C}{(n+1)^2}, \qquad \gamma^{u_2^I}(x_n^{\uparrow}) = \frac{C}{(n+1)^2} \qquad (n \geq 0).$$

*Proof.* Start from $(w, x^m, C)$ given by Lemma 24. Apply Lemma 25 recursively to the intervals

$$[x_{n+1}^{\downarrow}, x_n^{\downarrow}], \qquad n \geq 0,$$

always leaving the utility unchanged to the right of $x_n^{\downarrow}$. This produces a sequence $(w_n^L)_n$ of elements of $\mathcal{U}^I$ whose gap is weakly increasing on each interval already treated. The $n$th modification changes values by at most the endpoint gap on that interval, hence by at most $C/(n+1)^2$. Since $\sum_n C/(n+1)^2 < \infty$, the sequence is uniformly Cauchy on the compact set $\mathcal{D}_1^I \cup \mathcal{D}_2^I$ and converges uniformly to a continuous function $u^L$.

Exactly the same recursion on the right-hand intervals

$$[x_n^{\uparrow}, x_{n+1}^{\uparrow}], \qquad n \geq 0,$$

starting from $u^L$, produces a uniformly convergent sequence with continuous limit $u_2^I$. Every step is obtained by post-composing with a continuous strictly increasing map, so order is preserved at each stage; the limit therefore still represents $\succeq_2$ on the compact ordered set $\mathcal{D}_1^I \cup \mathcal{D}_2^I$. By construction, the gap of $u_2^I$ is weakly increasing to the left of $x^m$, weakly decreasing to the right, preserves the prescribed iterate values, and vanishes at the endpoints because the two branches are commonly indifferent there by Lemma 20. The gap stays strictly positive on the interior because no ironing step reverses the strict branch order. □

### C.3.2 The companion index for agent 1

Once the agent 2 gap is single-peaked, the companion index for agent 1 is obtained by summing those gaps along the counterpart chains.

**Lemma 27** (Fundamental balance construction on one interval)**.** *Let $u_2^I$ be as in Lemma 26, and write*

$$\gamma(x) := u_2^I(d_2^x) - u_2^I(d_1^x).$$

*Define $u_1^I : \mathcal{D}_1^I \cup \mathcal{D}_2^I \to \mathbb{R}$ by*

$$u_1^I(d_1^x) := u_2^I(d_1^{x_I}) + \sum_{n=0}^{\infty} \gamma\big(x_n^{\downarrow}(x)\big), \qquad x \in [x_I, x^m], \tag{11}$$

$$u_1^I(d_1^x) := u_2^I(d_1^{x^I}) - \sum_{n=0}^{\infty} \gamma\big(x_n^{\uparrow}(x)\big), \qquad x \in [x^m, x^I], \tag{12}$$

*and*

$$u_1^I(d_2^x) := u_1^I(d_1^{\eta(x)}), \qquad x \in I. \tag{13}$$

*Then the series converge absolutely, $u_1^I$ represents $\succeq_1$ on $\mathcal{D}_1^I \cup \mathcal{D}_2^I$, and*

$$u_1^I(d_1^x) - u_1^I(d_2^x) = u_2^I(d_2^x) - u_2^I(d_1^x) = \gamma(x) \qquad (x \in I).$$

*Proof.* Lemma 26 yields

$$\gamma\big(x_n^{\downarrow}(x)\big) \leq \frac{C}{(n+1)^2}, \qquad \gamma\big(x_n^{\uparrow}(x)\big) \leq \frac{C}{(n+1)^2},$$

for a common constant $C > 0$. Thus the two series converge absolutely.

For $x \in (x_I, x^m]$, the iterate chain from $\eta(x)$ is the tail of the iterate chain from $x$, so

$$x_n^{\downarrow}(\eta(x)) = x_{n+1}^{\downarrow}(x).$$

Subtracting the formulas for $u_1^I(d_1^x)$ and $u_1^I(d_2^x) = u_1^I(d_1^{\eta(x)})$ therefore leaves exactly the first term, namely $\gamma(x)$. The same argument on the right branch gives the same identity for $x \in [x^m, x^I)$, and continuity covers the endpoints.

It remains to verify that $u_1^I$ represents $\succeq_1$. If $x < y \leq x^m$, then $\eta$ is increasing, hence

$$x_n^{\downarrow}(y) > x_n^{\downarrow}(x) \qquad \text{for every } n.$$

Since $\gamma$ is weakly increasing on $[x_I, x^m]$ and strictly positive on the interior, every term in the series for $u_1^I(d_1^y)$ is at least the corresponding term for $u_1^I(d_1^x)$, and the first is strictly larger. Thus $u_1^I(d_1^y) > u_1^I(d_1^x)$. The same argument with $\theta$ and the weak decrease of $\gamma$ on $[x^m, x^I]$ shows that $u_1^I$ is strictly increasing along the $d_1$-branch in the order of $\succeq_1$.

On the $d_2$-branch, (13) and the strict monotonicity of $\eta$ imply the same conclusion. Finally, (10) identifies $d_2^x$ as the unique point on the opposite branch that is $\succeq_1$-indifferent to $d_1^{\eta(x)}$, and (13) assigns the same value to that pair. Hence the induced cross-branch comparisons also coincide with $\succeq_1$. □

## C.4 Balanced pairs and the global representation

We now pass from local interval constructions to the global boundary pair, and then from that pair to the representation in Theorem 1.

**Definition 15** (Balanced pair on the dilemma boundary)**.** *A pair of continuous functions*

$$m_1^2,\ m_2^1 : \mathcal{D}_1^{\succeq} \cup \mathcal{D}_2^{\succeq} \to \mathbb{R}$$

*is* balanced *if $m_1^2$ represents $\succeq_1$, $m_2^1$ represents $\succeq_2$, and, for every $x \in [0,1]$,*

$$m_1^2(d_1^x) - m_1^2(d_2^x) = m_2^1(d_2^x) - m_2^1(d_1^x). \tag{14}$$

The next lemma is the only general utility-matching fact we need. It replaces several longer transform arguments used in the original draft.

**Lemma 28** (Matching two continuous utilities on a compact ordered set)**.** *Let $C \subseteq F$ be compact. Let $u, v : C \to \mathbb{R}$ be continuous and suppose that, for all $h, h' \in C$,*

$$u(h) \geq u(h') \quad \Longleftrightarrow \quad v(h) \geq v(h').$$

*Then there exists a continuous strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$v = \phi \circ u \qquad \text{on } C.$$

*Proof.* Let $T := u(C)$. Since $C$ is compact and $u$ is continuous, $T$ is compact. Define $\phi_0 : T \to \mathbb{R}$ by $\phi_0(u(h)) := v(h)$. The hypothesis makes $\phi_0$ well defined and strictly increasing. Continuity follows from compactness: if $t_n \to t$ in $T$ and $u(h_n) = t_n$, then some subsequence of $(h_n)$ converges to $\bar{h} \in C$, so $u(\bar{h}) = t$ and hence $v(h_n) \to v(\bar{h}) = \phi_0(t)$. Extend $\phi_0$ to a continuous strictly increasing map on $\mathbb{R}$ by affine interpolation on the components of $\mathbb{R} \setminus T$. □

We first glue the local constructions into one balanced pair on the whole dilemma boundary.

**Lemma 29** (Global balanced pair under CLE)**.** *Assume $\succeq$ is pseudo-egalitarian and satisfies Axiom **??**. Then there exists a balanced pair $(m_1^2, m_2^1)$ on $\mathcal{D}_1^{\succeq} \cup \mathcal{D}_2^{\succeq}$.*

*Proof.* For each $I \in \mathcal{I}$, Lemma 21 fixes one of the two disagreement orientations on $(x_I, x^I)$. If the maintained orientation of the local subsection holds, Lemma 27 gives a pair $(u_1^I, u_2^I)$ on $\mathcal{D}_1^I \cup \mathcal{D}_2^I$ that represents $(\succeq_1, \succeq_2)$ and satisfies the balance identity on $I$. In the opposite orientation, apply the same construction after swapping the two individuals; this again produces a local balanced pair on $I$.

Distinct maximal disagreement intervals have disjoint interiors and can meet only at endpoints. By Lemma 20, both individuals are indifferent between the two branches at every common endpoint. Adding the same constant to both coordinates of a local pair preserves both the represented preferences and the balance identity, so we can normalize neighboring local pairs to match continuously at shared endpoints.

Now let $J = [a, b]$ be a connected component of the complement of the union of all disagreement interiors. On $J$, at least one individual is indifferent between $d_1^x$ and $d_2^x$ for each $x$, hence both are by Lemma 20. Therefore the two branches carry the same order for both individuals on $J$. Choose continuous strictly increasing functions

$$v_1^J,\ v_2^J : [a, b] \to \mathbb{R}$$

whose endpoint values match the already normalized neighboring pieces, and define

$$m_1^2(d_1^x) = m_1^2(d_2^x) := v_1^J(x), \qquad m_2^1(d_1^x) = m_2^1(d_2^x) := v_2^J(x) \qquad (x \in J).$$

On such a component, (14) holds trivially.

Doing this on every disagreement interval and every complementary agreement component yields globally defined continuous functions $(m_1^2, m_2^1)$ on $\mathcal{D}_1^{\succeq} \cup \mathcal{D}_2^{\succeq}$. By construction they represent $\succeq_1$ and $\succeq_2$ there and satisfy (14) for every $x \in [0, 1]$. □

**Lemma 30** (Uniqueness from single-recipient comparisons)**.** *Let $\succeq$ and $\succeq'$ be pseudo-egalitarian social preferences on $\mathcal{F}$. If, for all $f, g \in F$,*

$$(f, 0) \succeq (0, g) \quad \Longleftrightarrow \quad (f, 0) \succeq' (0, g),$$

*then $\succeq = \succeq'$ on $\mathcal{F}$.*

*Proof.* By Theorem 3, $\succeq$ and $\succeq'$ admit aligned leximin representations

$$\mathbf{m} = (m_1, m_2) \qquad \text{and} \qquad \mathbf{m}' = (m_1', m_2'),$$

where, for each $i \in \{1, 2\}$, both $m_i$ and $m_i'$ represent $\succeq_i$. Applying a common positive affine normalization to each pair if necessary, assume

$$m_1(0) = m_2(0) = m_1'(0) = m_2'(0) = 0, \qquad m_1(1) = m_2(1) = m_1'(1) = m_2'(1) = 1.$$

Since $F$ is convex and each $m_i, m_i'$ is continuous, every image $m_i(F)$ and $m_i'(F)$ is a connected subset of $\mathbb{R}$. Monotonicity and the normalization imply that each image contains 0 and 1 and lies in $[0, 1]$. Hence

$$m_i(F) = m_i'(F) = [0, 1] \qquad \text{for } i = 1, 2.$$

Because $m_i$ and $m_i'$ are continuous representations of the same preference $\succeq_i$, there is a strictly increasing homeomorphism $\alpha_i : [0, 1] \to [0, 1]$ such that

$$m_i' = \alpha_i \circ m_i.$$

We claim that $\alpha_1 = \alpha_2$. Fix $s, t \in [0, 1]$. Choose $f, g \in F$ with $m_1(f) = s$ and $m_2(g) = t$. Since $\mathbf{m}$ is aligned and normalized at 0,

$$(f, 0) \succeq (0, g) \quad \Longleftrightarrow \quad s \geq t.$$

Likewise,

$$(f, 0) \succeq' (0, g) \quad \Longleftrightarrow \quad \alpha_1(s) \geq \alpha_2(t).$$

By hypothesis,

$$\alpha_1(s) \geq \alpha_2(t) \quad \Longleftrightarrow \quad s \geq t \qquad \text{for all } s, t \in [0, 1].$$

Now fix $t \in [0, 1]$. Taking $s = t$ gives $\alpha_1(t) \geq \alpha_2(t)$. On the other hand, for every $s < t$

we have $\alpha_1(s) < \alpha_2(t)$. Letting $s \uparrow t$ and using continuity of $\alpha_1$ yields $\alpha_1(t) \leq \alpha_2(t)$. Thus $\alpha_1(t) = \alpha_2(t)$. Since $t$ was arbitrary, $\alpha_1 = \alpha_2 =: \phi$.

Therefore $m_i' = \phi \circ m_i$ for $i = 1, 2$. Because $\phi$ is strictly increasing, applying $\phi$ to both coordinates preserves the leximin order. Hence $\mathbf{m}$ and $\mathbf{m}'$ induce the same social preference on $\mathcal{F}$. Therefore $\succeq = \succeq'$. □

The next lemma shows that a balanced boundary pair already contains all the information needed for the representation theorem.

**Lemma 31** (From a balanced pair to a balanced Pareto-spined representation)**.** *Assume $\succeq$ is pseudo-egalitarian and let $(m_1^2, m_2^1)$ be a balanced pair on $\mathcal{D}_1^{\succeq} \cup \mathcal{D}_2^{\succeq}$. Then there exist Pareto-spined metrics*

$$\mathbf{m}^1 = (m_1^1, m_2^1) \qquad \textit{and} \qquad \mathbf{m}^2 = (m_1^2, m_2^2)$$

*such that the metric*

$$\mathbf{m} = (m_1, m_2), \qquad m_i := m_i^1 + m_i^2,\ i \in \{1, 2\},$$

*represents $\succeq$ through leximin. In particular, $\succeq$ admits a balanced Pareto-spined representation.*

*Proof.* Fix continuous utility representations $u_1, u_2$ of $(\succeq_1, \succeq_2)$.

First extend the balanced pair from the dilemma boundary to all of $F$. Since $m_1^2$ and $u_1$ represent the same order on the compact set $\mathcal{D}_1^{\succeq} \cup \mathcal{D}_2^{\succeq}$, Lemma 28 yields a continuous strictly increasing function $\phi_1$ such that

$$m_1^2 = \phi_1 \circ u_1 \qquad \text{on } \mathcal{D}_1^{\succeq} \cup \mathcal{D}_2^{\succeq}.$$

Extend $\phi_1$ to $\mathbb{R}$ and redefine $m_1^2 := \phi_1 \circ u_1$ on all of $F$. Do the same for $m_2^1$.

Next use Lemma 19, which identifies $\mathcal{D}_1^{\succeq}$ with $E_1$ and $\mathcal{D}_2^{\succeq}$ with $E_2$. Since $m_2^1|_{E_1}$ represents $\succeq_1$ on $E_1$, Lemma 28 yields a continuous strictly increasing transform of $u_1$ whose value coincides with $m_2^1$ on $E_1$; call the resulting global representation $m_1^1$. Then

$$m_1^1(h) = m_2^1(h) \qquad \text{for all } h \in E_1,$$

so $\mathbf{m}^1 = (m_1^1, m_2^1)$ is Pareto-spined on $E_1$. Symmetrically, there is a global representation $m_2^2$ of $\succeq_2$ such that

$$m_2^2(h) = m_1^2(h) \qquad \text{for all } h \in E_2,$$

so $\mathbf{m}^2 = (m_1^2, m_2^2)$ is Pareto-spined on $E_2$.

Define

$$m_i := m_i^1 + m_i^2, \qquad i \in \{1, 2\}.$$

Since each $m_i^1$ and $m_i^2$ represents $\succeq_i$, so does $m_i$.

For every $x \in [0, 1]$, the spine equalities on $E_1$ and $E_2$ together with the balance identity (14) imply

$$\begin{aligned} m_1(d_1^x) - m_2(d_2^x) &= \big(m_1^1(d_1^x) - m_2^1(d_2^x)\big) + \big(m_1^2(d_1^x) - m_2^2(d_2^x)\big) \\ &= \big(m_2^1(d_1^x) - m_2^1(d_2^x)\big) + \big(m_1^2(d_1^x) - m_1^2(d_2^x)\big) = 0. \end{aligned}$$

Hence, with

$$\lambda(x) := m_1(d_1^x) = m_2(d_2^x),$$

the map $\lambda$ is continuous and strictly increasing because each branch $x \mapsto d_i^x$ is strictly increasing and each $m_i$ represents $\succeq_i$.

Let $\succeq^m$ denote the leximin preference represented by $\mathbf{m}$. For every $f \in F$, continuity and strict monotonicity of $\succeq_1$ yield a unique $x_1(f) \in [0, 1]$ with $f \sim_1 d_1^{x_1(f)}$. Likewise, for every $g \in F$, there is a unique $x_2(g) \in [0, 1]$ with $g \sim_2 d_2^{x_2(g)}$. Because $\succeq$ is pseudo-egalitarian,

Appendix B and Lemma 14 imply

$$(f,0) \succeq (0,g) \quad \Longleftrightarrow \quad x_1(f) \geq x_2(g).$$

Exactly the same equivalence holds for $\succeq^m$, because

$$m_1(f) = m_1(d_1^{x_1(f)}) = \lambda(x_1(f)), \qquad m_2(g) = m_2(d_2^{x_2(g)}) = \lambda(x_2(g)),$$

and $\lambda$ is strictly increasing. Thus $\succeq$ and $\succeq^m$ agree on all single-recipient comparisons. Lemma 30 therefore yields $\succeq^m = \succeq$ on $\mathcal{F}$. □

**Lemma 32** (Balanced Pareto-spined leximin implies CLE)**.** *Let $\succeq$ admit a leximin representation through a balanced Pareto-spined metric*

$$\mathbf{m} = (m_1, m_2), \qquad m_i = m_i^1 + m_i^2,\ i \in \{1,2\},$$

*where $\mathbf{m}^1$ is Pareto-spined on $E_1$ and $\mathbf{m}^2$ is Pareto-spined on $E_2$. Then $\succeq$ is pseudo-egalitarian and satisfies Axiom* **??***.*

*Proof.* Because $0, 1 \in E_1 \cap E_2$, the spine equalities imply

$$m_1(0) = m_2(0) \qquad \text{and} \qquad m_1(1) = m_2(1).$$

Hence Theorem 3 implies that $\succeq$ is pseudo-egalitarian.

Fix $(f_1, g_2) \in E_1 \times E_2$ such that

$$f_1 \succeq_1 g_2 \qquad \text{and} \qquad f_1 \succeq_2 g_2.$$

Since $\mathbf{m}^1$ is Pareto-spined on $E_1$ and $m_2^1$ represents $\succeq_2$,

$$m_1^1(f_1) = m_2^1(f_1) \geq m_2^1(g_2).$$

Since $\mathbf{m}^2$ is Pareto-spined on $E_2$ and $m_1^2$ represents $\succeq_1$,

$$m_1^2(f_1) \geq m_1^2(g_2) = m_2^2(g_2).$$

Adding the two inequalities gives

$$m_1(f_1) \geq m_2(g_2).$$

If at least one of the two individual comparisons is strict, then at least one displayed inequality is strict, hence

$$m_1(f_1) > m_2(g_2).$$

Because $m_1(0) = m_2(0)$, the aligned leximin metric compares $(f_1, 0)$ and $(0, g_2)$ exactly through the active coordinates. Therefore

$$m_1(f_1) \geq m_2(g_2) \quad \Longrightarrow \quad (f_1, 0) \succeq (0, g_2),$$

and strict inequality implies

$$(f_1, 0) \succ (0, g_2).$$

This is Axiom **??**. The case $(i,j) = (2,1)$ is symmetric. □

*Proof of Theorem 1.* For the forward direction, assume $\succeq$ is pseudo-egalitarian and satisfies Axiom **??**. Lemma 29 yields a balanced pair on the dilemma boundary, and Lemma 31 upgrades it to a balanced Pareto-spined leximin representation of $\succeq$.

For the reverse direction, let $\succeq$ admit a balanced Pareto-spined leximin representation. Lemma 32 implies that $\succeq$ is pseudo-egalitarian and satisfies Axiom **??**. □

## C.5 Weak Consensus and weighted Pareto-spined metrics

This subsection proves Theorem **??**. The proof uses the intended reading of the second clause of Axiom **??**: the consistency requirement concerns *one-sided* dilemma indifferences. Thus, if some pair in $\mathcal{D}^{\succeq}$ is resolved by individual $k$'s indifference but not by the other individual's, then every pair in $\mathcal{D}^{\succeq}$ is resolved by individual $k$'s indifference. The weighted theorem then follows from a trichotomy. Either one-sided dilemma indifference is globally resolved by individual 1, which yields the endpoint case $\alpha = 0$; or it is globally resolved by individual 2, which yields $\alpha = 1$; or one-sided dilemma indifference never occurs, in which case Axiom **??** collapses to CLE and Theorem 1 applies.

We begin with a simple existence fact. It shows that a Pareto spine can always be placed on either frontier branch. The endpoint cases below therefore need only construct the active branch of the weighted metric.

**Lemma 33** (Existence of one-sided Pareto spines)**.** *For each $i \in \{1, 2\}$, there exists a Pareto-spined metric on $E_i$.*

*Proof.* Fix continuous utility representations $u_1, u_2$ of $(\succeq_1, \succeq_2)$. We treat $i = 1$; the case $i = 2$ is symmetric.

Under Assumption **??**, the branch $x \mapsto f_1^x$ parametrizes $E_1$ and is strictly increasing for both individual orders. Hence, for $h, h' \in E_1$,

$$h \succeq_1 h' \quad \Longleftrightarrow \quad h \succeq_2 h'.$$

So $u_1|_{E_1}$ and $u_2|_{E_1}$ represent the same order on the compact set $E_1$. Lemma 28 therefore yields a continuous strictly increasing map $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi \circ u_1 = u_2 \qquad \text{on } E_1.$$

Set

$$m_1^1 := \phi \circ u_1, \qquad m_2^1 := u_2.$$

Then each $m_j^1$ represents $\succeq_j$, and

$$m_1^1(h) = m_2^1(h) \qquad \text{for all } h \in E_1.$$

Thus $\mathbf{m}^1 = (m_1^1, m_2^1)$ is Pareto-spined on $E_1$. □

The next two lemmas treat the endpoint cases. If every dilemma pair is resolved by individual 1's indifference, then the aligned leximin metric from Theorem 3 is already spined on $E_2$. The case for individual 2 is symmetric.

**Lemma 34** (Endpoint representation on $E_2$)**.** *Assume $\succeq$ is pseudo-egalitarian and*

$$f_i \sim_1 g_j \qquad \textit{for every } (f_i, g_j) \in \mathcal{D}^{\succeq}.$$

*Then $\succeq$ admits a weighted Pareto-spined leximin representation with $\alpha = 0$.*

*Proof.* By Theorem 3, $\succeq$ admits an aligned leximin representation $\mathbf{m} = (m_1, m_2)$. By Lemma 19, $\mathcal{D}_1^{\succeq} = E_1$ and $\mathcal{D}_2^{\succeq} = E_2$, and there are homeomorphisms

$$x \longmapsto d_1^x \in E_1, \qquad x \longmapsto d_2^x \in E_2$$

such that

$$(d_1^x, 0) \sim (0, d_2^x) \qquad \text{for all } x \in [0, 1].$$

By the maintained hypothesis, also

$$d_1^x \sim_1 d_2^x \qquad \text{for all } x \in [0, 1].$$

Since $m_1$ represents $\succeq_1$,

$$m_1(d_1^x) = m_1(d_2^x).$$

Since $\mathbf{m}$ represents $\succeq$ through aligned leximin and $(d_1^x, 0) \sim (0, d_2^x)$,

$$m_1(d_1^x) = m_2(d_2^x).$$

Hence

$$m_1(d_2^x) = m_2(d_2^x) \qquad \text{for all } x \in [0,1].$$

Because $x \mapsto d_2^x$ parametrizes $E_2$, $\mathbf{m}$ is Pareto-spined on $E_2$.

Choose any Pareto-spined metric $\mathbf{m}^1$ on $E_1$, whose existence is guaranteed by Lemma 33. Then $\mathbf{m}$ is a weighted Pareto-spined metric with $\alpha = 0$, since

$$\mathbf{m} = 0 \cdot \mathbf{m}^1 + 1 \cdot \mathbf{m}.$$

It already represents $\succeq$ through leximin, so this is the required weighted Pareto-spined representation. □

**Lemma 35** (Endpoint representation on $E_1$)**.** *Assume $\succeq$ is pseudo-egalitarian and*

$$f_i \sim_2 g_j \qquad \textit{for every } (f_i, g_j) \in \mathcal{D}^{\succeq}.$$

*Then $\succeq$ admits a weighted Pareto-spined leximin representation with $\alpha = 1$.*

*Proof.* The argument is symmetric to Lemma 34. Let $\mathbf{m} = (m_1, m_2)$ be an aligned leximin representation from Theorem 3. For every $x \in [0,1]$, Lemma 19 gives $(d_1^x, 0) \sim (0, d_2^x)$, and the maintained hypothesis gives $d_1^x \sim_2 d_2^x$. Since $m_2$ represents $\succeq_2$,

$$m_2(d_1^x) = m_2(d_2^x),$$

while social indifference gives

$$m_1(d_1^x) = m_2(d_2^x).$$

Therefore

$$m_1(d_1^x) = m_2(d_1^x) \qquad \text{for all } x \in [0,1].$$

Because $x \mapsto d_1^x$ parametrizes $E_1$, $\mathbf{m}$ is Pareto-spined on $E_1$.

Choose any Pareto-spined metric $\mathbf{m}^2$ on $E_2$ using Lemma 33. Then $\mathbf{m}$ is a weighted Pareto-spined metric with $\alpha = 1$, since

$$\mathbf{m} = 1 \cdot \mathbf{m} + 0 \cdot \mathbf{m}^2.$$

Again $\mathbf{m}$ already represents $\succeq$ through leximin. □

If no one-sided dilemma indifference occurs, the weak axiom collapses to the stronger one used in Theorem 1.

**Lemma 36** (No one-sided dilemma indifference implies CLE)**.** *Assume $\succeq$ satisfies Axiom* **??***, and suppose there is no $(f_i, g_j) \in \mathcal{D}^{\succeq}$ such that*

$$f_i \sim_1 g_j \textit{ and } f_i \not\sim_2 g_j, \qquad \textit{or} \qquad f_i \sim_2 g_j \textit{ and } f_i \not\sim_1 g_j.$$

*Then $\succeq$ satisfies Axiom* **??***.*

*Proof.* The weak part of Axiom **??** is exactly the first clause of Axiom **??**. It remains to verify the strict part.

Fix distinct $i, j \in \{1, 2\}$ and $(f_i, g_j) \in E_i \times E_j$ such that

$$f_i \succeq_1 g_j, \qquad f_i \succeq_2 g_j,$$

with at least one strict comparison. By Axiom **??**,

$$(f_i, 0) \succeq (0, g_j).$$

If both individual comparisons are strict, Axiom **??** already gives

$$(f_i, 0) \succ (0, g_j).$$

So only the case of exactly one strict comparison remains. Suppose, for contradiction, that

$$(f_i, 0) \sim (0, g_j).$$

Then $(f_i, g_j) \in \mathcal{D}^{\succeq}$. Since exactly one individual comparison is strict, this pair exhibits one-sided dilemma indifference, contrary to the maintained hypothesis. Hence

$$(f_i, 0) \succ (0, g_j).$$

This is the strict part of Axiom **??**. □

We now turn to the converse implication. A weighted Pareto-spined metric always implies pseudo-egalitarianism, and it satisfies the weak consensus requirement. The consistency clause is generated endogenously: interior weights rule out one-sided dilemma indifference, while endpoint weights resolve every dilemma pair through the same frontier spine.

**Lemma 37** (Weighted Pareto-spined leximin implies Weak CLE)**.** *Let $\succeq$ admit a leximin representation through a weighted Pareto-spined metric*

$$\mathbf{m}^\alpha = (m_1^\alpha, m_2^\alpha), \qquad m_j^\alpha = \alpha m_j^1 + (1-\alpha) m_j^2, \qquad \alpha \in [0,1],$$

*where $\mathbf{m}^1$ is Pareto-spined on $E_1$ and $\mathbf{m}^2$ is Pareto-spined on $E_2$. Then $\succeq$ is pseudo-egalitarian and satisfies Axiom **??**, with the consistency clause understood in the one-sided sense stated at the beginning of this subsection.*

*Proof.* Because $0, 1 \in E_1 \cap E_2$, the spine equalities imply

$$m_1^\alpha(0) = m_2^\alpha(0) \qquad \text{and} \qquad m_1^\alpha(1) = m_2^\alpha(1).$$

Hence Theorem 3 implies that $\succeq$ is pseudo-egalitarian.

Fix $(f_1, g_2) \in E_1 \times E_2$ such that

$$f_1 \succeq_1 g_2 \qquad \text{and} \qquad f_1 \succeq_2 g_2.$$

Since $\mathbf{m}^1$ is Pareto-spined on $E_1$ and $m_2^1$ represents $\succeq_2$,

$$m_1^1(f_1) = m_2^1(f_1) \geq m_2^1(g_2).$$

Since $\mathbf{m}^2$ is Pareto-spined on $E_2$ and $m_1^2$ represents $\succeq_1$,

$$m_1^2(f_1) \geq m_1^2(g_2) = m_2^2(g_2).$$

Therefore

$$m_1^\alpha(f_1) \geq m_2^\alpha(g_2).$$

If both individual comparisons are strict, both displayed inequalities are strict, so

$$m_1^\alpha(f_1) > m_2^\alpha(g_2).$$

As in the proof of Lemma 32, aligned leximin therefore gives

$$(f_1, 0) \succeq (0, g_2),$$

with strict preference whenever both individual comparisons are strict. The case $(i, j) = (2, 1)$ is symmetric. This proves the first clause of Axiom **??**.

It remains to verify the consistency clause. Let $(f_1, g_2) \in \mathcal{D}^{\succeq}$, so

$$m_1^\alpha(f_1) = m_2^\alpha(g_2).$$

If $\alpha \in (0,1)$ and $f_1 \sim_1 g_2$, then

$$0 = m_1^\alpha(f_1) - m_2^\alpha(g_2) = \alpha\big(m_2^1(f_1) - m_2^1(g_2)\big) + (1-\alpha)\underbrace{\big(m_1^2(f_1) - m_1^2(g_2)\big)}_{=0},$$

so $m_2^1(f_1) = m_2^1(g_2)$ and hence $f_1 \sim_2 g_2$. Thus one-sided dilemma indifference is impossible when $\alpha \in (0,1)$.

If $\alpha = 0$, then every $(f_1, g_2) \in \mathcal{D}^{\succeq}$ satisfies

$$0 = m_1^2(f_1) - m_2^2(g_2) = m_1^2(f_1) - m_1^2(g_2),$$

so $f_1 \sim_1 g_2$. Hence every dilemma pair is resolved by individual 1's indifference. Similarly, if $\alpha = 1$, every dilemma pair is resolved by individual 2's indifference.

Therefore the one-sided consistency clause is satisfied in all cases. □

We can now prove the weighted characterization.

*Proof of Theorem* **??**. Assume first that $\succeq$ is pseudo-egalitarian and satisfies Axiom **??**, with the consistency clause interpreted in the one-sided sense stated above.

If there exists $(f_i, g_j) \in \mathcal{D}^{\succeq}$ such that

$$f_i \sim_1 g_j \qquad \text{and} \qquad f_i \not\sim_2 g_j,$$

then the consistency clause implies

$$f_i' \sim_1 g_j' \qquad \text{for every } (f_i', g_j') \in \mathcal{D}^{\succeq}.$$

Lemma 34 therefore yields a weighted Pareto-spined representation with $\alpha = 0$.

If instead there exists $(f_i, g_j) \in \mathcal{D}^{\succeq}$ such that

$$f_i \sim_2 g_j \qquad \text{and} \qquad f_i \not\sim_1 g_j,$$

the symmetric argument and Lemma 35 yield a weighted Pareto-spined representation with $\alpha = 1$.

If neither case occurs, then no one-sided dilemma indifference exists. Lemma 36 implies that $\succeq$ satisfies Axiom **??**. By Theorem 1, $\succeq$ admits a balanced Pareto-spined representation

$$\mathbf{m} = (m_1, m_2), \qquad m_j = m_j^1 + m_j^2.$$

Define

$$\widetilde{m}_j^1 := 2m_j^1, \qquad \widetilde{m}_j^2 := 2m_j^2, \qquad j \in \{1,2\}.$$

Then $\widetilde{\mathbf{m}}^1$ and $\widetilde{\mathbf{m}}^2$ are still Pareto-spined, and

$$m_j = \frac{1}{2}\,\widetilde{m}_j^1 + \frac{1}{2}\,\widetilde{m}_j^2.$$

So $\mathbf{m}$ is also a weighted Pareto-spined metric, with $\alpha = \frac{1}{2}$.

Conversely, if $\succeq$ admits a weighted Pareto-spined leximin representation, Lemma 37 implies that $\succeq$ is pseudo-egalitarian and satisfies Axiom **??**. This proves Theorem **??**. □

# D Proofs for Section ??: More than two agents

This appendix proves Theorem **??**. The argument is local and pairwise. It does not construct a single many-agent interpersonal scale. It shows instead that pseudo-egalitarianism identifies a common baseline for single-recipient comparisons, and that Axiom **??** then forces the two relevant claimant indices to coincide on a set large enough to contain every revealed $(i,j)$-dilemma. Unlike Appendix C, no frontier parametrization is needed here.

Throughout, let $N = \{1, \ldots, n\}$ with $n \geq 3$. Acts are measurable maps from $\Omega$ to $[0, 1]$, identified $\mu$-almost everywhere, and $F$ is endowed with the metric $d_m$ from Appendix A. We write $0, 1 \in F$ for the constant acts.

## D.1 Many-agent setup, normalization, and pairwise calibration

An allocation is a vector $\mathbf{f} = (f_1, \ldots, f_n) \in F^n$, and feasibility is imposed state by state:

$$\mathcal{F}^n \equiv \Big\{\mathbf{f} \in F^n : \sum_{i \in N} f_i(\omega) \leq 1 \text{ for } \mu\text{-almost every } \omega \in \Omega\Big\}.$$

We endow $F^n$ with the product metric

$$d_m^n(\mathbf{f}, \mathbf{g}) \equiv \max_{i \in N} d_m(f_i, g_i).$$

Let $E^n \subseteq \mathcal{F}^n$ denote the Pareto set, and for each $i \in N$ let

$$E_i^n \equiv \{x \in F : \exists f_{-i} \in F^{n-1} \text{ such that } (x, f_{-i}) \in E^n\}.$$

For $x \in F$ and $i \in N$, write $x^{i0} \in \mathcal{F}^n$ for the single-recipient allocation with $x$ in coordinate $i$ and 0 elsewhere. As in Section **??**, for distinct $i, j \in N$ define

$$D_{\succeq}^{ij} \equiv \{(x, y) \in E_i^n \times E_j^n : x^{i0} \sim y^{j0}\}.$$

Thus Axiom **??** says exactly that every revealed pairwise dilemma $(x, y) \in D_{\succeq}^{ij}$ must satisfy

$$x \sim_i y \qquad \text{and} \qquad x \sim_j y.$$

The first step is to normalize the common baseline created by Axiom **??**. This is the only normalization fact used later.

**Lemma 38** (Single-recipient symmetry yields common baselines)**.** *Let $\succeq$ be pseudo-egalitarian on $\mathcal{F}^n$, and let $\mathbf{m} = (m_1, \ldots, m_n)$ be a leximin-representing metric. Then*

$$m_i(0) = m_j(0) \qquad \textit{and} \qquad m_i(1) = m_j(1) \qquad \textit{for all } i, j \in N.$$

*Consequently, after a common strictly increasing transformation, one may normalize the metric so that*

$$m_k(0) = 0 \qquad \textit{and} \qquad m_k(1) = 1 \qquad \textit{for all } k \in N.$$

*Proof.* Fix distinct $i, j \in N$ and write $a_k := m_k(1)$ and $b_k := m_k(0)$ for $k \in N$. Since each $m_k$ represents $\succeq_k$ and $1 \succ_k 0$ by Assumption **??**, we have $a_k > b_k$ for every $k$.

The utility vector of $1^{i0}$ is

$$(b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_n),$$

and that of $1^{j0}$ is

$$(b_1, \ldots, b_{j-1}, a_j, b_{j+1}, \ldots, b_n).$$

Because $1^{i0} \sim 1^{j0}$ by Axiom **??**, these two vectors are leximin-indifferent, hence have the same sorted coordinates. Cancelling the common coordinates $\{b_k : k \notin \{i, j\}\}$ leaves

$$\{a_i, b_j\} = \{a_j, b_i\}$$

as multisets. Since $a_i > b_i$ and $a_j > b_j$, the only possible identification is

$$a_i = a_j \qquad \text{and} \qquad b_i = b_j.$$

Because $i$ and $j$ were arbitrary, the endpoint values are common across individuals. A common strictly increasing transformation then normalizes them to 0 and 1. □

The theorem is stated in terms of local pairwise coincidence sets.

**Definition 16** (Pairwise spines)**.** *Let $\succeq$ be pseudo-egalitarian with leximin-representing metric $\mathbf{m} = (m_1, \ldots, m_n)$. For distinct $i, j \in N$, a set $S^{ij} \subseteq F$ is an* $(i,j)$-spine *if*

$$m_i(h) = m_j(h) \qquad \text{for all } h \in S^{ij}.$$

*The metric* $\mathbf{m}$ *is* pairwise spined on dilemmas *if, for every distinct $i, j \in N$, there exists an $(i,j)$-spine $S^{ij}$ such that*

$$D^{ij}_{\succeq} \subseteq S^{ij} \times S^{ij}.$$

Once the baseline is normalized, a single-recipient social tie is exactly equality of the two active coordinates.

**Lemma 39** (Calibration via single-recipient dilemmas)**.** *Let $\succeq$ be pseudo-egalitarian and let $\mathbf{m} = (m_1, \ldots, m_n)$ be a leximin-representing metric normalized so that*

$$m_k(0) = 0 \qquad \text{for all } k \in N.$$

*Fix distinct $i, j \in N$ and acts $x, y \in F$. Then*

$$x^{i0} \sim y^{j0} \quad \Longleftrightarrow \quad m_i(x) = m_j(y).$$

*In particular, this equivalence holds for every $(x, y) \in E^n_i \times E^n_j$.*

*Proof.* Under the normalization, the utility vector of $x^{i0}$ has $n-1$ coordinates equal to 0 and one active coordinate $m_i(x)$. The corresponding vector for $y^{j0}$ has $n-1$ coordinates equal to 0 and one active coordinate $m_j(y)$. Monotonicity gives $m_i(x) \geq 0$ and $m_j(y) \geq 0$, so leximin compares the two vectors only through those active coordinates. Therefore

$$x^{i0} \sim y^{j0} \quad \Longleftrightarrow \quad m_i(x) = m_j(y).$$

□

## D.2 Characterization

The proof of Theorem **??** is now immediate. A revealed pairwise dilemma gives one cross-person equality by Lemma 39. Axiom **??** adds the two within-person equalities needed to place both acts on a common $(i,j)$-spine.

*Proof of Theorem* **??**. Let $\succeq$ be pseudo-egalitarian on $\mathcal{F}^n$, and let $\mathbf{m} = (m_1, \ldots, m_n)$ be a leximin-representing metric. By Lemma 38, after a common strictly increasing transformation we may assume

$$m_k(0) = 0 \qquad \text{for all } k \in N.$$

This transformation preserves both the representation and the pairwise-spine property.

For the forward direction, assume Axiom **??**. Fix distinct $i, j \in N$ and define

$$S^{ij} \equiv \{h \in F : m_i(h) = m_j(h)\}.$$

Then $S^{ij}$ is an $(i,j)$-spine. Let $(x, y) \in D^{ij}_{\succeq}$. Since $x^{i0} \sim y^{j0}$, Lemma 39 gives

$$m_i(x) = m_j(y).$$

Axiom **??** gives $x \sim_i y$ and $x \sim_j y$, hence

$$m_i(x) = m_i(y) \qquad \text{and} \qquad m_j(x) = m_j(y).$$

Combining these equalities yields

$$m_i(x) = m_j(x) \qquad \text{and} \qquad m_i(y) = m_j(y),$$

so $x, y \in S^{ij}$. Since $(x, y)$ was arbitrary, $\mathbf{m}$ is pairwise spined on dilemmas.

For the reverse direction, assume $\mathbf{m}$ is pairwise spined on dilemmas. Fix distinct $i, j \in N$ and $(x, y) \in D^{ij}_{\succeq}$. By hypothesis, there exists an $(i, j)$-spine $S^{ij}$ with $x, y \in S^{ij}$. Therefore

$$m_i(x) = m_j(x) \qquad \text{and} \qquad m_i(y) = m_j(y).$$

Because $(x, y) \in D^{ij}_{\succeq}$, Lemma 39 also gives

$$m_i(x) = m_j(y).$$

Hence

$$m_i(x) = m_i(y) \qquad \text{and} \qquad m_j(x) = m_j(y).$$

Since $m_i$ and $m_j$ represent $\succeq_i$ and $\succeq_j$, it follows that

$$x \sim_i y \qquad \text{and} \qquad x \sim_j y.$$

This is exactly Axiom **??**. □

## D.3 Two-agent bridge: common indifference on knife edges

This final subsection records the two-agent knife-edge analogue of Axiom **??**. It is not used in the proof of Theorem **??**. Its role is interpretive: it makes explicit that the many-agent axiom is the pairwise extension of a two-agent restriction stated only on the planner's own single-recipient dilemmas.

**Axiom 5** (Dilemma common indifference (DCI))**.** *For all $(f_1, g_2) \in E_1 \times E_2$,*

$$(f_1, 0) \sim (0, g_2) \quad \Longrightarrow \quad f_1 \sim_i g_2 \;\; \textit{for all } i \in \{1, 2\}.$$

Under pseudo-egalitarianism, DCI is exactly the two-agent knife-edge version of the pairwise-spine requirement.

**Proposition 13** (DCI and the two-agent knife-edge spine restriction)**.** *Assume $\succeq$ is pseudo-egalitarian. Then Axiom 5 holds if and only if $\succeq$ admits an aligned leximin representation through a metric $\mathbf{m} = (m_1, m_2)$ such that, with*

$$S \equiv \{h \in F : m_1(h) = m_2(h)\},$$

*every pair $(f_1, g_2) \in E_1 \times E_2$ satisfying*

$$(f_1, 0) \sim (0, g_2)$$

*also satisfies*

$$f_1 \in S \qquad \textit{and} \qquad g_2 \in S.$$

*Proof.* By Theorem 3, let $\mathbf{m} = (m_1, m_2)$ be an aligned leximin representation of $\succeq$.

For the forward direction, assume Axiom 5 and let $(f_1, g_2) \in E_1 \times E_2$ satisfy

$$(f_1, 0) \sim (0, g_2).$$

Lemma 14 gives

$$m_1(f_1) = m_2(g_2).$$

Axiom 5 gives $f_1 \sim_1 g_2$ and $f_1 \sim_2 g_2$, hence

$$m_1(f_1) = m_1(g_2) \qquad \text{and} \qquad m_2(f_1) = m_2(g_2).$$

Combining these equalities yields

$$m_1(f_1) = m_2(f_1) \qquad \text{and} \qquad m_1(g_2) = m_2(g_2),$$

so $f_1, g_2 \in S$.

For the reverse direction, assume the displayed spine restriction. Let $(f_1, g_2) \in E_1 \times E_2$ satisfy

$$(f_1, 0) \sim (0, g_2).$$

Then $f_1, g_2 \in S$, so

$$m_1(f_1) = m_2(f_1) \qquad \text{and} \qquad m_1(g_2) = m_2(g_2).$$

Lemma 14 also gives

$$m_1(f_1) = m_2(g_2).$$

Therefore

$$m_1(f_1) = m_1(g_2) \qquad \text{and} \qquad m_2(f_1) = m_2(g_2).$$

Since $m_1$ and $m_2$ represent $\succeq_1$ and $\succeq_2$, this implies

$$f_1 \sim_1 g_2 \qquad \text{and} \qquad f_1 \sim_2 g_2.$$

This is Axiom 5. □

The bridge to the many-agent theorem is now transparent. The two-agent knife edge

$$(f_1, 0) \sim (0, g_2)$$

is replaced by pairwise single-recipient dilemmas

$$x^{i0} \sim y^{j0},$$

and the conclusion that both claimants are indifferent between the two acts becomes exactly Axiom **??**. The many-agent result is therefore the pairwise extension of DCI, with Axiom **??** supplying the common baseline needed for those pairwise calibrations.

# E Proofs for Section ?? (A): envy-freeness off and on the frontier

This appendix supports Section **??**. It proves Proposition **??**, Proposition **??**, and Proposition **??**. The first result shows that a global binary-menu envy principle conflicts with pseudo-egalitarian leximin once the aligned coordinates genuinely differ. The second shows that, on the efficient frontier, envy-freeness reduces to a single threshold equality, which is exactly Assumption **??**. The third then shows that this threshold condition leaves the pseudo-egalitarian ordering underidentified. Proposition **??** is proved separately in Appendix F.

Throughout, maintain Assumption **??** and Assumption **??**. When proving Proposition **??** and Proposition **??**, also maintain the hypotheses of Proposition **??**. In particular, $\succeq$ is pseudo-egalitarian, admits a balanced Pareto-spined metric as in Theorem 1, and $EF \neq \varnothing$. By Theorem 3, $\succeq$ therefore has an aligned leximin representation $\mathbf{m} = (m_1, m_2)$. Normalize it by $m_1(0) = m_2(0) = 0$, and write $x \mapsto f^x = (f_1^x, f_2^x) \in E$, $x \in [0, 1]$, for the frontier parametrization from Assumption **??**, with $f^0 = (0, 1)$ and $f^1 = (1, 0)$.

## E.1 Global conditional envy-freeness fails

The main-text proposition already isolates the relevant configuration. The proof is immediate once one compares the leximin minima.

*Proof of Proposition* **??**. Let $f := (x, x)$ and $g := (y, z)$. Since both coordinates of $f$ coincide, $f \in EF^{\mathcal{F}}$.

Next, $g \notin EF^{\mathcal{F}}$. Indeed, $m_2(y) > m_2(z)$ and $m_2$ represents $\succeq_2$, so $y \succ_2 z$. Individual 2 therefore strictly prefers the other person's bundle at $g$.

The two allocations are Pareto-incomparable. Since $m_1(x) > m_1(y)$, one has $x \succ_1 y$, hence $f \succ_1 g$. Since $m_2(z) > m_2(x)$, one has $z \succ_2 x$, hence $g \succ_2 f$.

It remains to compare them socially. Under $\mathbf{m}$,

$$\mathbf{m}(f) = (m_1(x), m_2(x)) \qquad \text{and} \qquad \mathbf{m}(g) = (m_1(y), m_2(z)).$$

By assumption, $m_1(x) > m_2(x)$, $m_1(y) > m_2(x)$, and $m_2(z) > m_2(x)$. Hence the leximin minimum of $f$ is $m_2(x)$, while the leximin minimum of $g$ is strictly larger. Therefore $g \succ f$.

Thus $f$ is envy-free, $g$ is not, neither Pareto-dominates the other, and yet $g \succ f$. So conditional envy-freeness fails on $\{(x,x),(y,z)\}$. □

## E.2 The envy threshold on the efficient frontier

The proof of Proposition **??** is one-dimensional. Along the frontier, let

$$A(x) := m_1(f_1^x), \qquad B(x) := m_2(f_2^x), \qquad W(x) := \min\{A(x), B(x)\}.$$

Because $m_1$ represents $\succeq_1$, $m_2$ represents $\succeq_2$, and the frontier branches are monotone by Assumption **??**, $A$ is continuous and strictly increasing, while $B$ is continuous and strictly decreasing. Frontier leximin comparisons are governed first by $W$.

We also use the single-recipient comparison identity

$$(f_1, 0) \sim (0, g_2) \quad \Longleftrightarrow \quad m_1(f_1) = m_2(g_2), \tag{15}$$

which is Lemma 14 under the normalization $m_1(0) = m_2(0) = 0$.

The first step is to locate the two efficient envy thresholds.

**Lemma 40** (Envy thresholds)**.** *There exist unique $\underline{x}, \overline{x} \in [0,1]$ such that $\partial_1 EF = \{f^{\underline{x}}\}$ and $\partial_2 EF = \{f^{\overline{x}}\}$. Since $EF \neq \varnothing$, one has $\underline{x} \leq \overline{x}$, and*

$$EF = \{f^x : x \in [\underline{x}, \overline{x}]\}.$$

*Proof.* Let $U := \{x \in [0,1] : f_1^x \succeq_1 f_2^x\}$ and $V := \{x \in [0,1] : f_2^x \succeq_2 f_1^x\}$. Continuity of preferences and of the frontier map makes both sets closed.

If $x < y$ and $x \in U$, Assumption **??** gives $f_1^y \succeq_1 f_1^x \succeq_1 f_2^x \succeq_1 f_2^y$, so $y \in U$. Thus $U$ is an upper set. Since $f_1^1 = 1 \succ_1 0 = f_2^1$, $U$ is nonempty, hence $U = [\underline{x}, 1]$ for a unique $\underline{x} \in [0,1]$. Closedness of $U$ gives $f_1^{\underline{x}} \sim_1 f_2^{\underline{x}}$, so $\partial_1 EF = \{f^{\underline{x}}\}$.

Similarly, if $x < y$ and $y \in V$, then $f_2^x \succeq_2 f_2^y \succeq_2 f_1^y \succeq_2 f_1^x$, so $x \in V$. Thus $V$ is a lower set. Since $f_2^0 = 1 \succ_2 0 = f_1^0$, $V$ is nonempty, hence $V = [0, \overline{x}]$ for a unique $\overline{x} \in [0,1]$. Closedness gives $f_2^{\overline{x}} \sim_2 f_1^{\overline{x}}$, so $\partial_2 EF = \{f^{\overline{x}}\}$.

Finally, $EF = \{f^x : x \in U \cap V\}$. Since $U = [\underline{x}, 1]$ and $V = [0, \overline{x}]$, and since $EF \neq \varnothing$, one has $\underline{x} \leq \overline{x}$ and $EF = \{f^x : x \in [\underline{x}, \overline{x}]\}$. □

The proposition from the main text now becomes a statement about the values of $W$ at the two threshold points. Axiom **??** requires all frontier points in $[\underline{x}, \overline{x}]$ to dominate all frontier points outside that interval. Assumption **??**, by (15), requires exactly $A(\underline{x}) = B(\overline{x})$.

*Proof of Proposition* **??**. We first show that Axiom **??** implies

$$A(\underline{x}) \leq B(\underline{x}) \qquad \text{and} \qquad B(\overline{x}) \leq A(\overline{x}). \tag{16}$$

Suppose $A(\underline{x}) > B(\underline{x})$. Then $W(\underline{x}) = B(\underline{x})$. By continuity of $A$ and strict decrease of $B$, some $y < \underline{x}$ close enough to $\underline{x}$ satisfies $A(y) > B(\underline{x})$ and $B(y) > B(\underline{x})$, hence $W(y) > W(\underline{x})$. Since $f^{\underline{x}} \in EF$ and $f^y \notin EF$, this contradicts Axiom **??**. The second inequality is symmetric.

Assume next that Axiom **??** holds. We claim that $A(\underline{x}) = B(\overline{x})$. If $A(\underline{x}) < B(\overline{x})$, continuity and strict decrease of $B$ yield $y > \overline{x}$ with $A(\underline{x}) < B(y) < B(\overline{x})$. By (16), $A(\overline{x}) \geq B(\overline{x}) > B(y)$, so $W(y) = B(y) > A(\underline{x}) = W(\underline{x})$, again contradicting Axiom **??**. If $A(\underline{x}) > B(\overline{x})$, continuity and strict increase of $A$ yield $y < \underline{x}$ with $B(\overline{x}) < A(y) < A(\underline{x})$.

By (16), $B(\underline{x}) \geq A(\underline{x}) > A(y)$, so $W(y) = A(y) > B(\overline{x}) = W(\overline{x})$, which is the same contradiction. Hence

$$A(\underline{x}) = B(\overline{x}).$$

By (15), this is equivalent to $(f_1^{\underline{x}}, 0) \sim (0, f_2^{\overline{x}})$, hence to Assumption **??**.

Conversely, assume Assumption **??**. By (15),

$$A(\underline{x}) = B(\overline{x}) =: c.$$

For every $x \in [\underline{x}, \overline{x}]$, monotonicity gives $A(x) \geq c$ and $B(x) \geq c$, hence $W(x) \geq c$. If $y < \underline{x}$, then $A(y) < A(\underline{x}) = c$, so $W(y) \leq A(y) < c$. If $y > \overline{x}$, then $B(y) < B(\overline{x}) = c$, so $W(y) \leq B(y) < c$. Therefore $W(x) > W(y)$ for every $x \in [\underline{x}, \overline{x}]$ and every $y \notin [\underline{x}, \overline{x}]$. Since leximin compares frontier points first by $W$, this implies $f^x \succ f^y$ for every $f^x \in EF$ and $f^y \in E \setminus EF$. Thus Axiom **??** holds. □

### E.3 Why efficient-only envy-freeness does not identify the ordering

The previous proof shows that, under the maintained structure, efficient-frontier envy-freeness reduces to one threshold equality. The final proposition shows that one equality leaves a large family of aligned leximin metrics intact.

*Proof of Proposition* **??**. Let $f \in \partial_1 EF$ and $g \in \partial_2 EF$, and write $c := m_1(f_1) = m_2(g_2)$. This is well defined by Lemma 40 and Proposition **??**. Fix a strictly increasing homeomorphism $\phi : [0,1] \to [0,1]$ with $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(c) = c$, and let $\succeq^\phi$ be represented by leximin through $(m_1, \phi \circ m_2)$.

Because $\phi$ is strictly increasing, $\phi \circ m_2$ still represents $\succeq_2$, and $m_1$ still represents $\succeq_1$. The endpoint equalities imply that $(m_1, \phi \circ m_2)$ remains aligned, so Theorem 3 implies that $\succeq^\phi$ is pseudo-egalitarian. Since

$$m_1(f_1) = c = \phi(c) = \phi(m_2(g_2)),$$

the threshold equality from Proposition **??** is preserved. Another application of Proposition **??** therefore yields Axiom **??** for $\succeq^\phi$.

It remains to show that $\succeq^\phi \neq \succeq$ when $\phi \neq \mathrm{id}$. Choose $t \in [0,1]$ with $\phi(t) \neq t$. Since $F$ is convex and each $m_i$ is continuous, $m_i(F)$ is an interval containing 0 and 1, hence $m_i(F) = [0,1]$. So there exist acts $a, b \in F$ with $m_1(a) = t = m_2(b)$. By Lemma 14,

$$(a, 0) \sim (0, b)$$

under $\succeq$. Under $\succeq^\phi$, the same lemma gives

$$(a, 0) \succeq^\phi (0, b) \quad \Longleftrightarrow \quad m_1(a) \geq \phi(m_2(b)) \quad \Longleftrightarrow \quad t \geq \phi(t).$$

Since $\phi(t) \neq t$, this comparison is strict in one direction. Thus $(a, 0)$ and $(0, b)$ are no longer indifferent under $\succeq^\phi$, so $\succeq^\phi \neq \succeq$. □

# F Proof for Section ?? (B): Counterexamples and Proposition ??

This appendix proves Proposition **??**. Both counterexamples use the same one-state environment, so all of the relevant geometry can be read directly from scalars. The only moving part is the common normalization built into the leximin metric. We first record the frontier facts used in both examples, then construct one pseudo-egalitarian preference satisfying Axiom **??** but not Axiom **??**, and another satisfying Axiom **??** but not Axiom **??**.

## F.1 Common one-state environment

Let $\Omega = \{\omega\}$ with $\mu(\Omega) = 1$. Then every act is identified with a scalar in $[0, 1]$, the feasible set is

$$\mathcal{F} = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\},$$

and the Pareto frontier is

$$E = \{(x, 1 - x) : x \in [0, 1]\}.$$

Throughout this appendix, both individuals have the same strictly increasing preference on $[0, 1]$, represented by

$$u_1(x) = u_2(x) = x.$$

**Lemma 41** (Frontier facts in the one-state environment)**.** *In the one-state environment above,*

$$EF = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}.$$

*Moreover, if*

$$f = (a, 1 - a) \in E, \qquad g = (b, 1 - b) \in E, \qquad a > b,$$

*then $f$ favors individual 1 relative to $g$, the disadvantaged acts are $f_2 = 1 - a$ and $g_1 = b$, and the weak and strict antecedents of Axiom* **??** *for $(f, g)$ are respectively*

$$1 - a \geq b \qquad \textit{and} \qquad 1 - a > b.$$

*Proof.* For $(x, 1 - x) \in E$, envy-freeness requires

$$x \geq 1 - x \qquad \text{and} \qquad 1 - x \geq x,$$

hence $x = 1/2$.

Now fix $a > b$. Along the frontier, $f$ gives more to individual 1 and less to individual 2 than $g$, so the disadvantaged acts are $f_2 = 1 - a$ and $g_1 = b$. Since both individuals rank acts by the usual order on $[0, 1]$, the weak and strict CLE antecedents are exactly the weak and strict comparisons of $1 - a$ with $b$. □

**Lemma 42** (Aligned continuous leximin metrics generate pseudo-egalitarian preferences)**.** *Let $\mathbf{m} = (m_1, m_2)$, where each $m_i : [0, 1] \to \mathbb{R}$ is continuous and strictly increasing, with*

$$m_1(0) = m_2(0) \qquad \textit{and} \qquad m_1(1) = m_2(1).$$

*Then the social preference on $\mathcal{F}$ represented by leximin through $\mathbf{m}$ is continuous and pseudo-egalitarian.*

*Proof.* Each $m_i$ is a continuous utility representation of the common individual preference. The endpoint equalities therefore place the induced leximin metric in the aligned class covered by Theorem 3. Continuity is immediate from continuity of the coordinate maps. □

## F.2 Envy-freeness does not imply CLE

We first construct a pseudo-egalitarian social preference that makes the unique envy-free frontier allocation strictly best among efficient allocations, yet leaves one strict-CLE comparison unresolved.

**Proposition 14.** *There exists a pseudo-egalitarian social preference that satisfies Axiom* **??** *but violates Axiom* **??***.*

*Proof.* Define

$$m_1(x) = x$$

and

$$m_2(x) = \begin{cases} \frac{2}{3}x, & 0 \le x \le \frac{3}{10}, \\ \frac{3}{2}x - \frac{1}{4}, & \frac{3}{10} < x \le \frac{1}{2}, \\ \frac{2}{3}x + \frac{1}{6}, & \frac{1}{2} < x \le \frac{4}{5}, \\ \frac{3}{2}x - \frac{1}{2}, & \frac{4}{5} < x \le 1. \end{cases}$$

The four pieces agree at $x = 3/10$, $x = 1/2$, and $x = 4/5$, and each has positive slope. Thus $m_2$ is continuous and strictly increasing. Also,

$$m_1(0) = m_2(0) = 0 \qquad \text{and} \qquad m_1(1) = m_2(1) = 1.$$

Let $\succeq$ be the social preference represented on $\mathcal{F}$ by leximin through $\mathbf{m} = (m_1, m_2)$. By Lemma 42, $\succeq$ is continuous and pseudo-egalitarian.

We first verify Axiom **??**. By Lemma 41,

$$EF = \Big\{\Big(\frac{1}{2}, \frac{1}{2}\Big)\Big\}.$$

At $f^* = (1/2, 1/2)$,

$$m_1\Big(\frac{1}{2}\Big) = m_2\Big(\frac{1}{2}\Big) = \frac{1}{2},$$

so

$$\min\{\mathbf{m}(f^*)\} = \frac{1}{2}.$$

If $f = (x, 1-x) \in E \setminus EF$, then either $x < 1/2$, in which case

$$\min\{\mathbf{m}(f)\} \le m_1(x) = x < \frac{1}{2},$$

or $x > 1/2$, in which case $1 - x < 1/2$ and strict monotonicity of $m_2$ gives

$$\min\{\mathbf{m}(f)\} \le m_2(1-x) < m_2\Big(\frac{1}{2}\Big) = \frac{1}{2}.$$

Hence every efficient allocation outside $EF$ has a strictly smaller leximin minimum than $f^*$, so $f^*$ is strictly best on $E$. This is exactly Axiom **??**.

Now consider

$$f = \Big(\frac{7}{10}, \frac{3}{10}\Big), \qquad g = \Big(\frac{1}{5}, \frac{4}{5}\Big).$$

Since $7/10 > 1/5$, Lemma 41 applies with

$$a = \frac{7}{10}, \qquad b = \frac{1}{5}.$$

The strict CLE antecedent is

$$1 - a > b, \qquad \text{that is,} \qquad \frac{3}{10} > \frac{1}{5},$$

so Axiom **??** would require $f \succ g$.

But

$$\mathbf{m}(f) = \Big(\frac{7}{10}, \frac{1}{5}\Big), \qquad \mathbf{m}(g) = \Big(\frac{1}{5}, \frac{7}{10}\Big),$$

because

$$m_2\Big(\frac{3}{10}\Big) = \frac{1}{5} \qquad \text{and} \qquad m_2\Big(\frac{4}{5}\Big) = \frac{7}{10}.$$

The two vectors have the same minimum and the same maximum, so leximin yields

$$f \sim g.$$

Thus Axiom **??** fails. □

## F.3 CLE does not imply envy-freeness

We next construct a pseudo-egalitarian social preference that satisfies every frontier comparison required by Axiom **??**, yet ranks a non-envy-free efficient allocation above the unique envy-free one.

**Proposition 15.** *There exists a pseudo-egalitarian social preference that satisfies Axiom* **??** *but violates Axiom* **??**.

*Proof.* Fix $\gamma \in (0,1)$ and define

$$m_1(x) = x, \qquad m_2(x) = x^\gamma \qquad (x \in [0,1]).$$

Both functions are continuous and strictly increasing, with

$$m_1(0) = m_2(0) = 0 \qquad \text{and} \qquad m_1(1) = m_2(1) = 1.$$

Let $\succeq$ be the leximin social preference induced by $\mathbf{m} = (m_1, m_2)$ on $\mathcal{F}$. By Lemma 42, $\succeq$ is continuous and pseudo-egalitarian.

We first verify Axiom **??**. Let

$$f = (a, 1-a), \qquad g = (b, 1-b)$$

be efficient allocations with $a > b$. By Lemma 41, the weak and strict CLE antecedents are

$$1 - a \geq b \qquad \text{and} \qquad 1 - a > b.$$

Assume first that $1 - a \geq b$. Since $a > b$,

$$m_1(a) = a > b.$$

Also, because $\gamma \in (0,1)$, one has $t^\gamma \geq t$ on $[0,1]$, so

$$m_2(1-a) = (1-a)^\gamma \geq 1 - a \geq b.$$

Hence

$$\min\{\mathbf{m}(f)\} \geq b.$$

For $g$, the first coordinate equals $b$, so

$$\min\{\mathbf{m}(g)\} \leq b.$$

Thus

$$\min\{\mathbf{m}(f)\} \geq \min\{\mathbf{m}(g)\}.$$

If the inequality is strict, then leximin gives $f \succ g$, hence $f \succeq g$. If instead the minima are equal, then necessarily

$$\min\{\mathbf{m}(f)\} = b = \min\{\mathbf{m}(g)\}.$$

Since $a > b$, equality on the left forces

$$(1-a)^\gamma = b.$$

But $1 - a \geq b$ and $t^\gamma > t$ for every $t \in (0,1)$, so this can happen only if

$$1 - a = b \in \{0, 1\}.$$

Because $a > b$, the only possibility is

$$a = 1, \qquad b = 0.$$

Then

$$f = (1, 0), \qquad g = (0, 1),$$

and

$$\mathbf{m}(f) = (1, 0), \qquad \mathbf{m}(g) = (0, 1),$$

so leximin yields $f \sim g$. Therefore the weak CLE conclusion holds in all cases.

Now assume the strict antecedent $1 - a > b$. Then

$$m_1(a) = a > b \qquad \text{and} \qquad m_2(1 - a) = (1 - a)^\gamma > 1 - a > b,$$

so

$$\min\{\mathbf{m}(f)\} > b \geq \min\{\mathbf{m}(g)\}.$$

Hence leximin yields $f \succ g$. This is exactly Axiom **??**.

To show that Axiom **??** fails, take

$$\gamma = \frac{1}{10}, \qquad f^* = \Big(\frac{1}{2}, \frac{1}{2}\Big) \in EF, \qquad g = \Big(\frac{9}{10}, \frac{1}{10}\Big) \in E \setminus EF.$$

By Lemma 41,

$$EF = \Big\{\Big(\frac{1}{2}, \frac{1}{2}\Big)\Big\}.$$

At $f^*$,

$$\min\{\mathbf{m}(f^*)\} = \min\Big\{\frac{1}{2}, \Big(\frac{1}{2}\Big)^{1/10}\Big\} = \frac{1}{2}.$$

At $g$,

$$\min\{\mathbf{m}(g)\} = \min\Big\{\frac{9}{10}, \Big(\frac{1}{10}\Big)^{1/10}\Big\} = \Big(\frac{1}{10}\Big)^{1/10}.$$

Since $10 < 2^{10}$,

$$\Big(\frac{1}{10}\Big)^{1/10} > \frac{1}{2}.$$

Therefore

$$\min\{\mathbf{m}(g)\} > \min\{\mathbf{m}(f^*)\},$$

so leximin ranks

$$g \succ f^*.$$

But $f^* \in EF$ and $g \in E \setminus EF$, so Axiom **??** fails. □

# G Proofs for Section ??: spines, equivalents, grading

This appendix proves the extension results from Section **??**. The common mechanism is simple. Once a one-dimensional comparison spine carries a common leximin scale, any socially indifferent knife edge on that spine equalizes the corresponding within-person equivalents. The appendix proceeds in three steps. It first shows that the CLE construction extends from the Pareto frontier to any ex ante specified monotone antichain. It then applies that observation to constant acts and extremal lotteries. It closes with the two grading propositions.

## G.1 Arbitrary comparison spines

We begin with the modular version of the boundary argument from Appendix C.

**Definition 17** (Allocation spine)**.** *A set $\Sigma \subseteq \mathcal{F}$ is an* allocation spine *if it is $d_m^2$-connected, contains $(1,0)$ and $(0,1)$, and is a Pareto antichain: for any distinct $f, g \in \Sigma$, neither $f$ Pareto-dominates $g$ nor $g$ Pareto-dominates $f$.*

**Assumption 4** (Parametrization of an allocation spine)**.** *Let $\Sigma$ be an allocation spine. There exists a $d_m^2$-continuous map*

$$x \longmapsto \sigma^x \in \Sigma, \qquad x \in [0,1],$$

*such that*

$$\sigma^0 = (0,1), \qquad \sigma^1 = (1,0),$$

*and for all $x > y$,*

$$\sigma_1^x \succ_i \sigma_1^y \qquad \textit{and} \qquad \sigma_2^y \succ_i \sigma_2^x \qquad \textit{for both } i \in \{1,2\}.$$

Write

$$\Sigma_i := \{h \in F : \exists k \in F \text{ such that } (h,k) \in \Sigma\}, \qquad i \in \{1,2\},$$

and define the induced dilemma set

$$\mathcal{D}^{\succeq}(\Sigma) := \big\{(f_1, g_2) \in \Sigma_1 \times \Sigma_2 : (f_1, 0) \sim (0, g_2)\big\}.$$

As in Section **??**, let CLE($\Sigma$) denote the restriction of Axiom **??** to dilemma comparisons in $\Sigma \times \Sigma$.

For a leximin metric $\mathbf{m} = (m_1, m_2)$, call a set $S \subseteq F$ a *spine* if

$$m_1(h) = m_2(h) \qquad \text{for all } h \in S.$$

The first lemma closes the small gap in the converse direction. Under pseudo-egalitarianism, every coordinate that appears on the spine also appears in some single-recipient dilemma pair on that spine.

**Lemma 43** (Every spine coordinate appears in a spine dilemma)**.** *Let $\succeq$ be pseudo-egalitarian and let $\Sigma$ satisfy Assumption 4. Then every act in $\Sigma_1 \cup \Sigma_2$ appears in some pair in $\mathcal{D}^{\succeq}(\Sigma)$.*

*Proof.* By Theorem 3, let $\mathbf{m} = (m_1, m_2)$ be an aligned leximin representation of $\succeq$. Define

$$a(x) := m_1(\sigma_1^x), \qquad b(x) := m_2(\sigma_2^x).$$

Assumption 4 implies that $a$ is continuous and strictly increasing, while $b$ is continuous and strictly decreasing. Since

$$\sigma^0 = (0,1) \qquad \text{and} \qquad \sigma^1 = (1,0),$$

alignment gives

$$a(0) = m_1(0) = m_2(0) = b(1), \qquad a(1) = m_1(1) = m_2(1) = b(0).$$

Hence the ranges of $a$ and $b$ coincide.

Fix $x \in [0,1]$. Since $a(x)$ lies in the range of $b$, there exists a unique $y \in [0,1]$ such that

$$a(x) = b(y).$$

By Lemma 14,

$$(\sigma_1^x, 0) \sim (0, \sigma_2^y),$$

so $(\sigma_1^x, \sigma_2^y) \in \mathcal{D}^{\succeq}(\Sigma)$. This shows that every act in $\Sigma_1$ appears in the first coordinate of some pair in $\mathcal{D}^{\succeq}(\Sigma)$. The same argument, swapping the roles of $a$ and $b$, treats $\Sigma_2$. □

*Proof of Proposition* **??**. Assume first that $\succeq$ is pseudo-egalitarian and satisfies CLE($\Sigma$). The proof of Appendix C uses only four properties of the comparison domain: it is a con-

nected one-dimensional antichain, it has a continuous parametrization with fixed endpoints, both coordinate branches are monotone in a common direction for both individuals, and single-recipient comparisons along the two branches can therefore be calibrated by continuity. Definition 17 and Assumption 4 provide exactly those ingredients. Repeating the argument of Appendix C with

$$E \rightsquigarrow \Sigma, \qquad E_i \rightsquigarrow \Sigma_i, \qquad \mathcal{D}^{\succeq} \rightsquigarrow \mathcal{D}^{\succeq}(\Sigma)$$

therefore yields a representative of the aligned leximin class whose common spine contains every act that appears in $\mathcal{D}^{\succeq}(\Sigma)$.

Conversely, suppose $\succeq$ admits a leximin representation through $\mathbf{m} = (m_1, m_2)$ and a spine $S$ containing every act that appears in $\mathcal{D}^{\succeq}(\Sigma)$. By Lemma 43,

$$\Sigma_1 \cup \Sigma_2 \subseteq S.$$

Fix a dilemma comparison in $\Sigma \times \Sigma$, say $f, g \in \Sigma$ with $f$ favoring individual 1 relative to $g$. Thus the disadvantaged acts are $f_2$ under $f$ and $g_1$ under $g$.

Suppose first that the weak CLE($\Sigma$) antecedent holds:

$$f_2 \succeq_1 g_1 \qquad \text{and} \qquad f_2 \succeq_2 g_1.$$

Since $f_2, g_1 \in S$,

$$m_2(f_2) = m_1(f_2) \qquad \text{and} \qquad m_1(g_1) = m_2(g_1).$$

Because $m_1$ and $m_2$ represent $\succeq_1$ and $\succeq_2$, the antecedent implies

$$m_2(f_2) \geq m_1(g_1).$$

Assumption 4 also gives

$$f_1 \succeq_1 g_1 \qquad \text{and} \qquad g_2 \succeq_2 f_2,$$

hence

$$m_1(f_1) \geq m_1(g_1) \qquad \text{and} \qquad m_2(g_2) \geq m_2(f_2).$$

So the first leximin comparison between $f$ and $g$ is exactly between

$$m_2(f_2) \qquad \text{and} \qquad m_1(g_1),$$

and the former is weakly larger. Therefore $f \succeq g$.

If one of the two individual comparisons in the antecedent is strict, then

$$m_2(f_2) > m_1(g_1),$$

so the same argument gives $f \succ g$. This is CLE($\Sigma$). □

## G.2 Constant acts and certainty equivalents

We now specialize the spine argument to the one-dimensional safe-act spine. This gives the certainty-equivalent corollary.

For each $c \in [0, 1]$, let $c$ also denote the constant act equal to $c$.

**Lemma 44** (Knife edges equalize unique spine equivalents)**.** *Let $\succeq$ be pseudo-egalitarian and let $\mathbf{m} = (m_1, m_2)$ be a leximin metric representing $\succeq$, normalized so that*

$$m_1(0) = m_2(0).$$

*Let $S \subseteq F$ be a spine for $\mathbf{m}$, and suppose $m_1$ is injective on $S$. If, for each $i \in \{1, 2\}$ and each $f \in F$, there exists a unique act $eq_i^S(f) \in S$ such that*

$$f \sim_i eq_i^S(f),$$

*then every* $(f_1, g_2) \in \mathcal{D}^{\succeq}$ *satisfies*

$$eq_1^S(f_1) = eq_2^S(g_2).$$

*Proof.* Fix $(f_1, g_2) \in \mathcal{D}^{\succeq}$. By definition,

$$(f_1, 0) \sim (0, g_2),$$

so Lemma 14 gives

$$m_1(f_1) = m_2(g_2).$$

Since $eq_1^S(f_1) \sim_1 f_1$ and $eq_2^S(g_2) \sim_2 g_2$,

$$m_1\big(eq_1^S(f_1)\big) = m_1(f_1) \qquad \text{and} \qquad m_2\big(eq_2^S(g_2)\big) = m_2(g_2).$$

Because both equivalents lie in $S$,

$$m_2\big(eq_2^S(g_2)\big) = m_1\big(eq_2^S(g_2)\big).$$

Hence

$$m_1\big(eq_1^S(f_1)\big) = m_1\big(eq_2^S(g_2)\big),$$

and injectivity of $m_1$ on $S$ yields the claim. □

**Definition 18** (Certainty-equivalent set)**.** *For* $i \in \{1, 2\}$ *and* $f \in F$*, let*

$$CE_i(f) := \{c \in [0, 1] : f \sim_i c\}.$$

**Lemma 45** (Existence and uniqueness of certainty equivalents)**.** *Assume* $\succeq_i$ *is* $d_m$*-continuous and monotone. Then* $CE_i(f)$ *is a nonempty closed interval for every* $f \in F$*. If, in addition, constants are strictly ranked,*

$$c > d \quad \Longrightarrow \quad c \succ_i d \qquad (c, d \in [0, 1]),$$

*then* $CE_i(f)$ *is a singleton for every* $f$*.*

*Proof.* Fix $f \in F$ and define

$$U := \{c \in [0, 1] : c \succeq_i f\}, \qquad L := \{c \in [0, 1] : f \succeq_i c\}.$$

Monotonicity gives $1 \in U$ and $0 \in L$, so both sets are nonempty. Continuity gives closedness. Since $U$ is an upper set and $L$ is a lower set on $[0, 1]$, the numbers

$$\bar{c} := \min U, \qquad \underline{c} := \max L$$

are well defined. If $\underline{c} < \bar{c}$, any $c \in (\underline{c}, \bar{c})$ belongs to either $U$ or $L$ by completeness, contradicting maximality or minimality. Hence $\underline{c} = \bar{c} =: c^*$, and

$$c^* \sim_i f.$$

Thus $CE_i(f) = U \cap L$ is nonempty and closed. If $c_1 < c_2$ both belong to $CE_i(f)$, then every $c \in [c_1, c_2]$ is also indifferent to $f$ by monotonicity and transitivity, so $CE_i(f)$ is an interval. Under strict monotonicity on constants, that interval must be a singleton. □

**Proposition 16** (Safe-act CLE yields a constant spine)**.** *Define*

$$\Sigma^{\text{safe}} := \{(c, 1 - c) : c \in [0, 1]\} \subseteq \mathcal{F},$$

*where* $c$ *denotes the constant act equal to* $c$*. Let* $\text{CLE}^{\text{safe}}$ *denote* CLE *restricted to dilemma comparisons in* $\Sigma^{\text{safe}} \times \Sigma^{\text{safe}}$*. If* $\succeq$ *is pseudo-egalitarian and satisfies* $\text{CLE}^{\text{safe}}$*, then there*

*exists a leximin metric* $\mathbf{m} = (m_1, m_2)$ *representing* $\succeq$ *such that*

$$m_1(c) = m_2(c) \qquad \text{for all } c \in [0,1].$$

*Proof.* The set $\Sigma^{\text{safe}}$ is an allocation spine satisfying Assumption 4, with parametrization

$$x \longmapsto (x, 1-x).$$

By Proposition **??**, it is enough to show that every constant act appears in $\mathcal{D}^{\succeq}(\Sigma^{\text{safe}})$. But

$$\Sigma_1^{\text{safe}} = \Sigma_2^{\text{safe}} = [0,1],$$

so this is exactly Lemma 43. The resulting spine therefore contains every constant act. □

**Corollary 2** (Safe-act CLE equalizes certainty equivalents)**.** *Assume* $\succeq$ *is pseudo-egalitarian and satisfies* $\text{CLE}^{\text{safe}}$*. Assume also that each* $CE_i(f)$ *is a singleton, and write* $ce_i(f)$ *for its unique element. Then every* $(f_1, g_2) \in \mathcal{D}^{\succeq}$ *satisfies*

$$ce_1(f_1) = ce_2(g_2).$$

*Proof.* By Proposition 16, there exists a leximin metric $\mathbf{m} = (m_1, m_2)$ such that all constants form a spine:

$$m_1(c) = m_2(c) \qquad \text{for all } c \in [0,1].$$

By strict monotonicity on constants, $m_1$ is injective on $[0,1]$. Apply Lemma 44 with

$$S = [0,1] \qquad \text{and} \qquad eq_i^S(f) = ce_i(f).$$

□

## G.3 Extremal lotteries and lottery equivalents

We now record the exact analogue for the spine of extremal lotteries in an Anscombe–Aumann environment (Anscombe and Aumann, 1963).

Let $X$ be a compact metric space of prizes, let $\Delta(X)$ be the set of Borel probability measures on $X$ with the weak topology, and fix benchmark prizes $x^+, x^- \in X$. For $p \in [0,1]$, define

$$\ell(p) := p\,\delta_{x^+} + (1-p)\,\delta_{x^-},$$

and let

$$\mathcal{L} := \{\ell(p) : p \in [0,1]\}.$$

Let $\widehat{F}$ be the set of measurable acts $\hat{f} : \Omega \to \Delta(X)$, and write $\ell(p)$ also for the constant act equal to that lottery.

**Assumption 5** (Regularity on the extremal-lottery spine)**.** *For each* $i \in \{1,2\}$*, the restriction of* $\succeq_i$ *to* $\mathcal{L}$ *is continuous and monotone in* $p$*, and*

$$\ell(1) \succeq_i \hat{f} \succeq_i \ell(0) \qquad \text{for every } \hat{f} \in \widehat{F}.$$

**Definition 19** (Lottery-equivalent set)**.** *For* $i \in \{1,2\}$ *and* $\hat{f} \in \widehat{F}$*, let*

$$LE_i(\hat{f}) := \{p \in [0,1] : \hat{f} \sim_i \ell(p)\}.$$

**Lemma 46** (Existence and uniqueness of lottery equivalents)**.** *Under Assumption 5,* $LE_i(\hat{f})$ *is a nonempty closed interval for every* $\hat{f} \in \widehat{F}$*. If, in addition,*

$$p > q \quad \Longrightarrow \quad \ell(p) \succ_i \ell(q),$$

*then* $LE_i(\hat{f})$ *is a singleton for every* $\hat{f}$*.*

*Proof.* The proof is identical to Lemma 45 after replacing constants $c$ by extremal lotteries $\ell(p)$. □

When $LE_i(\hat{f})$ is a singleton, write $le_i(\hat{f})$ for its unique element.

**Lemma 47** (Normalized expected-utility interpretation)**.** *Suppose $\succeq_i$ on $\widehat{F}$ admits an Anscombe–Aumann expected-utility representation (Anscombe and Aumann, 1963). Normalize the von Neumann–Morgenstern index so that*

$$u_i(x^-) = 0, \qquad u_i(x^+) = 1.$$

*Then*

$$\hat{f} \sim_i \ell(p) \quad \Longleftrightarrow \quad p = V_i(\hat{f}),$$

*where $V_i$ is the normalized expected-utility value of $\hat{f}$. In particular, if lottery equivalents are unique, then*

$$le_i(\hat{f}) = V_i(\hat{f}) \qquad \text{for all } \hat{f} \in \widehat{F}.$$

*Proof.* Under the Anscombe–Aumann representation, $V_i(\ell(p)) = p$ for every $p \in [0,1]$. Hence

$$\hat{f} \sim_i \ell(p) \quad \Longleftrightarrow \quad V_i(\hat{f}) = V_i(\ell(p)) \quad \Longleftrightarrow \quad V_i(\hat{f}) = p.$$

The final statement is immediate when the equivalent is unique. □

**Proposition 17** (Safe-lottery CLE yields the extremal-lottery spine)**.** *Let $\widehat{\mathcal{F}} \subseteq \widehat{F}^2$ contain every allocation of the form*

$$(\ell(p), \ell(1-p)), \qquad p \in [0,1].$$

*Let* $\mathrm{CLE}^{\Delta}$ *denote* CLE *restricted to these safe-lottery comparisons. If $\widehat{\succeq}$ is pseudo-egalitarian on $\widehat{\mathcal{F}}$ and satisfies* $\mathrm{CLE}^{\Delta}$*, then there exists a leximin metric $(\hat{m}_1, \hat{m}_2)$ representing $\widehat{\succeq}$ such that*

$$\hat{m}_1(\ell(p)) = \hat{m}_2(\ell(p)) \qquad \text{for all } p \in [0,1].$$

*Proof.* This is the exact analogue of Proposition 16. The comparison set

$$\widehat{\Sigma}^{\Delta} := \{(\ell(p), \ell(1-p)) : p \in [0,1]\}$$

is a one-dimensional antichain with continuous monotone parametrization

$$p \longmapsto (\ell(p), \ell(1-p)).$$

The proof of Proposition **??** therefore applies verbatim on the domain $\widehat{\mathcal{F}}$, and the analogue of Lemma 43 shows that every extremal lottery appears in an induced dilemma pair. Hence the resulting common spine contains all of $\mathcal{L}$. □

**Corollary 3** (Safe-lottery CLE equalizes lottery equivalents)**.** *Let $\widehat{\mathcal{F}} \subseteq \widehat{F}^2$ contain all single-recipient allocations $(\hat{f}, \ell(0))$ and $(\ell(0), \hat{g})$, as well as all safe-lottery allocations $(\ell(p), \ell(1-p))$. Assume $\widehat{\succeq}$ is pseudo-egalitarian on $\widehat{\mathcal{F}}$ and satisfies* $\mathrm{CLE}^{\Delta}$*. Assume also that lottery equivalents are unique for both individuals. Then every*

$$(\hat{f}_1, \hat{g}_2) \in \widehat{\mathcal{D}}^{\widehat{\succeq}} := \left\{(\hat{f}_1, \hat{g}_2) \in \widehat{F}^2 : (\hat{f}_1, \ell(0)) \sim_{\widehat{\succeq}} (\ell(0), \hat{g}_2)\right\}$$

*satisfies*

$$le_1(\hat{f}_1) = le_2(\hat{g}_2).$$

*Proof.* By Proposition 17, there exists a leximin metric $(\hat{m}_1, \hat{m}_2)$ such that

$$\hat{m}_1(\ell(p)) = \hat{m}_2(\ell(p)) \qquad \text{for all } p \in [0,1].$$

Strict monotonicity on the extremal-lottery spine makes $\hat{m}_1$ injective on $\mathcal{L}$. The proof of Lemma 44 now applies verbatim with $F$ replaced by $\widehat{F}$, the parked baseline 0 replaced by $\ell(0)$, and

$$S = \mathcal{L}, \qquad eq_i^S(\hat{f}) = le_i(\hat{f}).$$

□

### G.4 Boundary grading refinements

We now prove the two optional grading results. Throughout, let $\succeq$ be pseudo-egalitarian and let $\mathcal{D}^{\succeq}$ be the dilemma boundary from Definition **??**.

*Proof of Proposition* **??**. By Theorem 3, $\succeq$ admits an aligned leximin representation $\mathbf{m} = (m_1, m_2)$. For every $(f_1, g_2) \in \mathcal{D}^{\succeq}$, Lemma 14 gives

$$m_1(f_1) = m_2(g_2).$$

Hence the condition

$$m_1(f_1) = m_2(g_2) \quad \Longleftrightarrow \quad f_1 \sim_i g_2 \qquad \text{for every } (f_1, g_2) \in \mathcal{D}^{\succeq}$$

reduces simply to

$$f_1 \sim_i g_2 \qquad \text{for every } (f_1, g_2) \in \mathcal{D}^{\succeq},$$

which is exactly Boundary grading through $i$. □

*Proof of Proposition* **??**. Fix $(f_1, g_2) \in \mathcal{D}^{\succeq}$. Then

$$(f_1, 0) \sim (0, g_2).$$

If

$$f_1 \succeq_k g_2 \text{ for both } k \in \{1, 2\},$$

with strict preference for at least one $k$, then Axiom **??** applied to the single-recipient comparison implies

$$(f_1, 0) \succ (0, g_2),$$

contradicting $(f_1, 0) \sim (0, g_2)$. The symmetric case with

$$g_2 \succeq_k f_1 \text{ for both } k \in \{1, 2\},$$

strict for at least one $k$, is identical. This is exactly Proposition **??**. □

# H Berlin example: frontier and envy-free region

This appendix supplies the calculations behind Section 9. It first derives the efficient frontier and its monotone parametrization. It then identifies the envy-free segment of that frontier. It closes by recording the exact points and bundle comparisons used in the text.

A feasible allocation assigns West and East the bundles

$$(f_w^b, f_w^p) \qquad \text{and} \qquad (f_e^b, f_e^p),$$

where the superscripts $b$ and $p$ denote budget and administrative control. Feasibility requires

$$f_w^b + f_e^b \leq 1, \qquad f_w^p + f_e^p \leq 1. \tag{17}$$

Preferences are represented by

$$\nu_w(f^b, f^p) = (f^b)^{2/3}(f^p)^{1/3}, \qquad \nu_e(f^b, f^p) = (f^b)^{1/3}(f^p)^{2/3}. \tag{18}$$

## H.1 Efficient frontier

Because both utilities are strictly increasing in both goods, every efficient allocation exhausts both aggregate constraints. Interior efficient allocations are then characterized by tangency. On the boundary, only the two corners survive.

**Lemma 48** (Efficient allocations)**.** *Every Pareto-efficient allocation exhausts both constraints in* (17)*. Every interior Pareto-efficient allocation can be written uniquely as*

$$\mathbf{f}(p) = \big((b(p), p), (1 - b(p), 1 - p)\big), \qquad b(p) = \frac{4p}{1 + 3p}, \qquad p \in (0, 1). \tag{19}$$

*Conversely, every allocation of the form* (19) *is Pareto efficient. The only boundary efficient allocations are*

$$((0, 0), (1, 1)) \qquad \textit{and} \qquad ((1, 1), (0, 0)),$$

*and they are obtained as the limits*

$$\lim_{p \downarrow 0} \mathbf{f}(p) = ((0, 0), (1, 1)), \qquad \lim_{p \uparrow 1} \mathbf{f}(p) = ((1, 1), (0, 0)).$$

*Moreover,* $p \mapsto b(p)$ *is strictly increasing on* $[0, 1]$*.*

*Proof.* If $f_w^b + f_e^b < 1$, one can increase either person's budget share slightly without violating feasibility, and both utilities rise by (18). So every efficient allocation satisfies

$$f_w^b + f_e^b = 1.$$

The same argument gives

$$f_w^p + f_e^p = 1.$$

Now consider an interior efficient allocation. Write

$$p = f_w^p \in (0, 1), \qquad b = f_w^b \in (0, 1),$$

so full use gives

$$f_e^p = 1 - p, \qquad f_e^b = 1 - b.$$

Since both utilities are differentiable, strictly increasing, and strictly quasi-concave on $(0, \infty)^2$, interior Pareto efficiency is equivalent to equality of the two marginal rates of substitution. For West,

$$\frac{\partial \nu_w / \partial f^b}{\partial \nu_w / \partial f^p} = 2 \frac{f^p}{f^b},$$

and for East,

$$\frac{\partial \nu_e / \partial f^b}{\partial \nu_e / \partial f^p} = \frac{1}{2} \frac{f^p}{f^b}.$$

Evaluating at $(b, p)$ and $(1 - b, 1 - p)$ yields

$$2 \frac{p}{b} = \frac{1}{2} \frac{1 - p}{1 - b}, \qquad \text{equivalently} \qquad \frac{b}{1 - b} = 4 \frac{p}{1 - p}. \tag{20}$$

Solving (20) gives

$$b(p) = \frac{4p}{1 + 3p},$$

which proves the claimed parametrization. Conversely, every interior allocation of the form (19) satisfies feasibility, full use, and the tangency condition, hence is Pareto efficient.

It remains to treat boundary points. Suppose West receives zero of at least one good. If West still receives a positive amount of the other good, then West's utility is zero while a small transfer of that positive good from West to East leaves West's utility unchanged and strictly raises East's utility. Hence any efficient allocation with either $f_w^b = 0$ or $f_w^p = 0$

must satisfy

$$f_w^b = f_w^p = 0.$$

By full use, this gives the corner

$$((0,0),(1,1)).$$

The symmetric argument shows that any efficient allocation with either $f_e^b = 0$ or $f_e^p = 0$ must be

$$((1,1),(0,0)).$$

So these are the only boundary efficient allocations.

Finally,

$$\lim_{p\downarrow 0} b(p) = 0, \qquad \lim_{p\uparrow 1} b(p) = 1,$$

and

$$b'(p) = \frac{4}{(1+3p)^2} > 0 \qquad \text{for all } p \in [0,1].$$

This gives the endpoint limits and monotonicity. □

## H.2 Envy-free interval

Along the frontier, West receives $(b(p), p)$ and East receives $(1-b(p), 1-p)$. Envy-freeness can therefore be reduced to two scalar inequalities along the parameter $p$.

**Lemma 49** (Envy-free interval)**.** *Let $b(p)$ and $\mathbf{f}(p)$ be as in Lemma 48. Define*

$$Q_w(p) = 17p^3 - 3p^2 + 3p - 1, \qquad Q_e(p) = 5p^3 - 3p^2 + 3p - 1. \tag{21}$$

*For every $p \in (0,1)$,*

$$\nu_w\big(b(p), p\big) \geq \nu_w\big(1-b(p), 1-p\big) \quad \Longleftrightarrow \quad Q_w(p) \geq 0, \tag{22}$$

*and*

$$\nu_e\big(1-b(p), 1-p\big) \geq \nu_e\big(b(p), p\big) \quad \Longleftrightarrow \quad Q_e(p) \leq 0. \tag{23}$$

*Moreover, $Q_w$ and $Q_e$ are strictly increasing on $[0,1]$, each has a unique root in $(0,1)$, denoted $p^w$ and $p^e$, and*

$$p^w < p^e.$$

*Consequently,*

$$\mathbf{f}(p) \in EF \quad \Longleftrightarrow \quad p \in [p^w, p^e]. \tag{24}$$

*In particular,*

$$\frac{1}{5} < p^w < p^e < \frac{9}{20},$$

*and numerically*

$$p^w \approx 0.2841036534, \qquad p^e \approx 0.3864882096. \tag{25}$$

*Proof.* Since

$$b(p) = \frac{4p}{1+3p}, \qquad 1-b(p) = \frac{1-p}{1+3p},$$

West's no-envy condition is

$$\nu_w\big(b(p), p\big) \geq \nu_w\big(1-b(p), 1-p\big).$$

Cubing both sides and substituting gives

$$b(p)^2 p \geq (1-b(p))^2(1-p) \quad \Longleftrightarrow \quad 16p^3 - (1-p)^3 \geq 0,$$

which is exactly

$$Q_w(p) \geq 0.$$

This proves (22).

Similarly, East's no-envy condition is

$$\nu_e\big(1-b(p),1-p\big) \geq \nu_e\big(b(p),p\big).$$

Cubing both sides and substituting yields

$$(1-b(p))(1-p)^2 \geq b(p)p^2 \quad \Longleftrightarrow \quad (1-p)^3 - 4p^3 \geq 0,$$

which is equivalent to

$$Q_e(p) \leq 0.$$

This proves (23).

For monotonicity,

$$Q_w'(p) = 51p^2 - 6p + 3 = 3(17p^2 - 2p + 1), \qquad Q_e'(p) = 15p^2 - 6p + 3 = 3(5p^2 - 2p + 1).$$

Both quadratic factors have negative discriminant, so both derivatives are strictly positive on $\mathbb{R}$. Hence $Q_w$ and $Q_e$ are strictly increasing. Since

$$Q_w(0) = Q_e(0) = -1, \qquad Q_w(1) = 16, \qquad Q_e(1) = 4,$$

each polynomial has a unique root in $(0,1)$.

Also,

$$Q_w(p) - Q_e(p) = 12p^3 > 0 \qquad \text{for } p > 0.$$

So at the unique root $p^w$ of $Q_w$,

$$Q_e(p^w) < 0,$$

which implies

$$p^w < p^e.$$

To obtain the bounds used in the text, note that

$$Q_w\Big(\frac{1}{5}\Big) = -\frac{48}{125} < 0, \qquad Q_e\Big(\frac{1}{5}\Big) = -\frac{12}{25} < 0,$$

and

$$Q_w\Big(\frac{9}{20}\Big) = \frac{10333}{8000} > 0, \qquad Q_e\Big(\frac{9}{20}\Big) = \frac{317}{1600} > 0.$$

Strict monotonicity therefore yields

$$\frac{1}{5} < p^w < p^e < \frac{9}{20}.$$

Finally, envy-freeness is the conjunction of (22) and (23). Since $Q_w$ is strictly increasing and vanishes at $p^w$,

$$Q_w(p) \geq 0 \quad \Longleftrightarrow \quad p \geq p^w,$$

and since $Q_e$ is strictly increasing and vanishes at $p^e$,

$$Q_e(p) \leq 0 \quad \Longleftrightarrow \quad p \leq p^e.$$

Combining the two proves (24). The numerical values in (25) are the corresponding unique roots. □

## H.3 Specific frontier points and bundle comparisons

The main text uses three explicit frontier points and two comparisons between their disadvantaged bundles. The next lemma records those values exactly.

**Lemma 50** (Named frontier points and illustrative comparisons)**.** *The frontier points used in Section 9 are*

$$\mathbf{f}(0.55) = \left( \left( \frac{44}{53}, \frac{11}{20} \right), \left( \frac{9}{53}, \frac{9}{20} \right) \right),$$

$$\mathbf{f}(0.45) = \left( \left( \frac{36}{47}, \frac{9}{20} \right), \left( \frac{11}{47}, \frac{11}{20} \right) \right),$$

*and*

$$\mathbf{f}(0.20) = \left( \left( \frac{1}{2}, \frac{1}{5} \right), \left( \frac{1}{2}, \frac{4}{5} \right) \right).$$

*Moreover,*

$$0.20 < p^w < p^e < 0.45 < 0.55,$$

*so all three lie outside the envy-free interval, with $p = 0.20$ to its left and $p = 0.45, 0.55$ to its right.*

*In the comparison between $\mathbf{f}(0.55)$ and $\mathbf{f}(0.45)$, the disadvantaged bundles are*

$$a = \left( \frac{9}{53}, \frac{9}{20} \right) \qquad \text{and} \qquad c = \left( \frac{36}{47}, \frac{9}{20} \right),$$

*and both West and East strictly prefer $c$ to $a$.*

*In the comparison between $\mathbf{f}(0.55)$ and $\mathbf{f}(0.20)$, the disadvantaged bundles are*

$$a = \left( \frac{9}{53}, \frac{9}{20} \right) \qquad \text{and} \qquad b = \left( \frac{1}{2}, \frac{1}{5} \right),$$

*West strictly prefers $b$ to $a$, while East strictly prefers $a$ to $b$.*

*Proof.* The exact coordinates follow from Lemma 48. Since

$$b(0.55) = \frac{4 \cdot 11/20}{1 + 3 \cdot 11/20} = \frac{44}{53}, \qquad b(0.45) = \frac{4 \cdot 9/20}{1 + 3 \cdot 9/20} = \frac{36}{47}, \qquad b(0.20) = \frac{4/5}{1 + 3/5} = \frac{1}{2},$$

the three frontier points are exactly those stated.

By Lemma 49,

$$\frac{1}{5} < p^w < p^e < \frac{9}{20} < \frac{11}{20},$$

which is

$$0.20 < p^w < p^e < 0.45 < 0.55.$$

Since $p \mapsto b(p)$ is strictly increasing, the frontier moves monotonically toward West as $p$ rises. So $\mathbf{f}(0.55)$ favors West relative to both $\mathbf{f}(0.45)$ and $\mathbf{f}(0.20)$, which makes the disadvantaged bundles exactly those listed.

For the comparison between $\mathbf{f}(0.55)$ and $\mathbf{f}(0.45)$, the bundles

$$a = \left( \frac{9}{53}, \frac{9}{20} \right) \qquad \text{and} \qquad c = \left( \frac{36}{47}, \frac{9}{20} \right)$$

have the same control level and

$$\frac{36}{47} > \frac{9}{53}.$$

Since both utilities in (18) are strictly increasing in budget holding control fixed, both individuals strictly prefer $c$ to $a$.

For the comparison between $\mathbf{f}(0.55)$ and $\mathbf{f}(0.20)$, compare cubes of utilities. For West,

$$\nu_w(f^b, f^p)^3 = (f^b)^2 f^p,$$

so

$$\nu_w(b)^3 = \Big(\frac{1}{2}\Big)^2 \frac{1}{5} = \frac{1}{20}, \qquad \nu_w(a)^3 = \Big(\frac{9}{53}\Big)^2 \frac{9}{20} = \frac{729}{56180}.$$

Since

$$\frac{1}{20} = \frac{2809}{56180} > \frac{729}{56180},$$

West strictly prefers $b$ to $a$.

For East,

$$\nu_e(f^b, f^p)^3 = f^b (f^p)^2,$$

so

$$\nu_e(b)^3 = \frac{1}{2}\Big(\frac{1}{5}\Big)^2 = \frac{1}{50}, \qquad \nu_e(a)^3 = \frac{9}{53}\Big(\frac{9}{20}\Big)^2 = \frac{729}{21200}.$$

Since

$$\frac{1}{50} = \frac{424}{21200} < \frac{729}{21200},$$

East strictly prefers $a$ to $b$. □

## H.4 A linear Berlin variant with explicit benchmark spines

This subsection records a variant of the Berlin example in which the two local spines can be written down in closed form. It is not used in the formal characterizations. Its role is to make the benchmark-spine construction from Section 2 completely explicit.

Let East and West consume budget and administrative control, written in the same order as above:

$$(b_e, p_e) \qquad \text{and} \qquad (b_w, p_w).$$

Feasibility requires

$$b_e + b_w \le 1, \qquad p_e + p_w \le 1.$$

Assume linear utilities

$$u_e(b,p) = \beta_e b + \rho_e p, \qquad u_w(b,p) = \beta_w b + \rho_w p,$$

with

$$\beta_e, \beta_w, \rho_e, \rho_w > 0 \qquad \text{and} \qquad \frac{\rho_e}{\beta_e} > \frac{\rho_w}{\beta_w}. \tag{26}$$

Condition (26) says that East values control more relative to budget than West does.

**Lemma 51** (The efficient set is L-shaped)**.** *Under* (26)*, the Pareto set is*

$$E^L = \Big\{((0,p),(1,1-p)) : p \in [0,1]\Big\} \cup \Big\{((b,1),(1-b,0)) : b \in [0,1]\Big\}.$$

*Hence the East and West coordinate projections are*

$$E^L_e = \{(0,p) : p \in [0,1]\} \cup \{(b,1) : b \in [0,1]\},$$

$$E^L_w = \{(1,p) : p \in [0,1]\} \cup \{(b,0) : b \in [0,1]\}.$$

*Proof.* Write an allocation as $((b,p),(1-b,1-p))$ with $(b,p) \in [0,1]^2$. For $\lambda \in [0,1]$, the weighted utilitarian objective is

$$\lambda u_e(b,p) + (1-\lambda) u_w(1-b,1-p) = (1-\lambda)(\beta_w + \rho_w) + \big(\lambda\beta_e - (1-\lambda)\beta_w\big)b + \big(\lambda\rho_e - (1-\lambda)\rho_w\big)p.$$

Since the objective is linear and the feasible set is the square $[0,1]^2$, every Pareto-efficient allocation maximizes this expression for some $\lambda \in [0,1]$, and every maximizer is Pareto efficient.

Let

$$\lambda_b := \frac{\beta_w}{\beta_e + \beta_w}, \qquad \lambda_p := \frac{\rho_w}{\rho_e + \rho_w}.$$

Condition (26) is equivalent to $\lambda_p < \lambda_b$. Thus: if $\lambda < \lambda_p$, both coefficients are negative and the unique maximizer is $(b, p) = (0, 0)$; if $\lambda = \lambda_p$, the coefficient on $b$ is negative and the coefficient on $p$ is zero, so the maximizers are exactly $(0, p)$ with $p \in [0, 1]$; if $\lambda_p < \lambda < \lambda_b$, the coefficient on $b$ is negative and the coefficient on $p$ is positive, so the unique maximizer is $(0, 1)$; if $\lambda = \lambda_b$, the coefficient on $b$ is zero and the coefficient on $p$ is positive, so the maximizers are exactly $(b, 1)$ with $b \in [0, 1]$; if $\lambda > \lambda_b$, both coefficients are positive and the unique maximizer is $(1, 1)$.

The union of these maximizers is exactly

$$\{(0, p) : p \in [0, 1]\} \cup \{(b, 1) : b \in [0, 1]\}$$

for East's bundle, which gives the stated L-shaped efficient set. The formulas for the coordinate projections are immediate. □

The next proposition constructs the two explicit benchmark spines. The East-spined metric coincides with East's utility on $E_e^L$, while the West-spined metric coincides with West's utility on $E_w^L$.

Define

$$sc_e(t) := \begin{cases} \dfrac{\rho_e}{\rho_w}\, t, & 0 \le t \le \rho_w, \\ \dfrac{\beta_e}{\beta_w}\, t + \rho_e - \dfrac{\beta_e \rho_w}{\beta_w}, & \rho_w \le t \le \rho_w + \beta_w, \end{cases}$$

and

$$sc_w(t) := \begin{cases} \dfrac{\beta_w}{\beta_e}\, t, & 0 \le t \le \beta_e, \\ \dfrac{\rho_w}{\rho_e}\, t + \beta_w - \dfrac{\rho_w \beta_e}{\rho_e}, & \beta_e \le t \le \beta_e + \rho_e. \end{cases}$$

**Proposition 18** (Explicit East-spined and West-spined metrics)**.** *Under* (26)*, $sc_e$ and $sc_w$ are continuous and strictly increasing. Moreover,*

$$\mathbf{m}^e := (m_e^e, m_w^e) := \big(u_e,\, sc_e \circ u_w\big)$$

*is Pareto-spined on $E_e^L$, and*

$$\mathbf{m}^w := (m_e^w, m_w^w) := \big(sc_w \circ u_e,\, u_w\big)$$

*is Pareto-spined on $E_w^L$.*

*Proof.* The two formulas are continuous at their kinks because

$$\frac{\rho_e}{\rho_w}\rho_w = \rho_e \qquad \text{and} \qquad \frac{\beta_e}{\beta_w}\rho_w + \rho_e - \frac{\beta_e \rho_w}{\beta_w} = \rho_e,$$

and likewise

$$\frac{\beta_w}{\beta_e}\beta_e = \beta_w \qquad \text{and} \qquad \frac{\rho_w}{\rho_e}\beta_e + \beta_w - \frac{\rho_w \beta_e}{\rho_e} = \beta_w.$$

Both slopes are strictly positive, so both maps are strictly increasing.

Since $sc_e$ and $sc_w$ are strictly increasing, $m_w^e = sc_e \circ u_w$ represents $\succeq_w$ and $m_e^w = sc_w \circ u_e$ represents $\succeq_e$. Thus each pair $\mathbf{m}^e$ and $\mathbf{m}^w$ represents the corresponding individual preferences.

Now take $h \in E_e^L$. If $h = (0, p)$, then

$$u_e(h) = \rho_e p \qquad \text{and} \qquad u_w(h) = \rho_w p \le \rho_w,$$

so

$$m_w^e(h) = sc_e(u_w(h)) = \frac{\rho_e}{\rho_w}\rho_w p = \rho_e p = u_e(h) = m_e^e(h).$$

If $h = (b, 1)$, then

$$u_e(h) = \beta_e b + \rho_e \qquad \text{and} \qquad u_w(h) = \beta_w b + \rho_w \geq \rho_w,$$

so

$$m_w^e(h) = sc_e(u_w(h)) = \frac{\beta_e}{\beta_w}(\beta_w b + \rho_w) + \rho_e - \frac{\beta_e \rho_w}{\beta_w} = \beta_e b + \rho_e = u_e(h) = m_e^e(h).$$

Hence $\mathbf{m}^e$ is Pareto-spined on $E_e^L$.

The proof for $\mathbf{m}^w$ is symmetric. If $h = (b, 0) \in E_w^L$, then

$$u_w(h) = \beta_w b, \qquad u_e(h) = \beta_e b,$$

so

$$m_e^w(h) = sc_w(u_e(h)) = \frac{\beta_w}{\beta_e}\beta_e b = \beta_w b = u_w(h) = m_w^w(h).$$

If $h = (1, p) \in E_w^L$, then

$$u_w(h) = \beta_w + \rho_w p \qquad \text{and} \qquad u_e(h) = \beta_e + \rho_e p,$$

so

$$m_e^w(h) = sc_w(u_e(h)) = \frac{\rho_w}{\rho_e}(\beta_e + \rho_e p) + \beta_w - \frac{\rho_w \beta_e}{\rho_e} = \beta_w + \rho_w p = u_w(h) = m_w^w(h).$$

Thus $\mathbf{m}^w$ is Pareto-spined on $E_w^L$. □

The two local spines can then be combined exactly as in Definition 2.

**Corollary 4** (An explicit balanced Pareto-spined metric)**.** *Under* (26)*, the metric*

$$\mathbf{m} := \mathbf{m}^e + \mathbf{m}^w$$

*is balanced Pareto-spined. Equivalently,*

$$m_e = u_e + sc_w \circ u_e, \qquad m_w = u_w + sc_e \circ u_w$$

*is the sum of an East-spined metric and a West-spined metric.*

*Proof.* Proposition 18 shows that $\mathbf{m}^e$ is Pareto-spined on $E_e^L$ and $\mathbf{m}^w$ is Pareto-spined on $E_w^L$. The conclusion is therefore immediate from Definition 2. □

This linear variant makes the benchmark-spine construction completely explicit. The efficient set is kinked, the two scaling maps are piecewise linear, and the balanced metric is obtained by summing the two benchmark spines in closed form.