Online Appendix to "Consensus and Interpersonal Welfare Comparisons"

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This online appendix supplies the supporting results required for the main paper. It establishes the reference-path and continuous-representation facts, population consistency, and the common-reference results used in the policy application. It also verifies the running example, gives sufficient conditions for regularity, and proves the reference-choice and institutional boundaries discussed in the main text. Proofs of the principal characterization, envy, minimal-reference, and policy results remain in the main paper. The two documents together contain the statements and proofs used by the submitted argument.

O.1. BILATERAL REFERENCE PATHS AND THE CONSENSUS RESTRICTION

The main representation uses single-recipient welfare positions and each person's indifferent bundle on either efficient path. We first verify that these objects exist and are continuous, then give the equivalent no-forbidden-tie formulation of EC. All notation and the standing assumptions are as in Sections 2–3 of the main paper.

Lemma O.1. *If $\succsim$ is represented by an interpersonal welfare scale and an aggregation rule, then for all $x, y \in X$,*

$$\mathbf{e}_1(x) \succsim \mathbf{e}_2(y) \quad \Longleftrightarrow \quad m_1(x) \geq m_2(y).$$

Proof of Lemma O.1. Write $a = m_1(x)$ and $b = m_2(y)$; giving x only to town 1 yields $(a, 0)$; giving y only to town 2 yields $(0, b)$. By single-recipient anonymity the second profile is socially tied to $(b, 0)$. Strong Pareto then compares the two numbers: the first allocation is weakly preferred exactly when $a \geq b$, and strictly preferred when $a > b$. □

The regular-path assumption gives a unique physical representative on either reference path, even when the two people disagree elsewhere.

Proposition O.1. *Under Assumption 1, every indifference class of either agent meets each efficient projection exactly once. Equivalently, if $v_i : X \to [0, 1]$ is any continuous normalized representation of $\succsim_i$, then*

$$v_i|_{E_j} : E_j \to [0, 1]$$

is a homeomorphism for every $i, j \in \{1, 2\}$.

Proof. Fix person i and reference path E_j; the parameterization $p_j(t)$ moves continuously and strictly upward in i's ranking as t rises from 0 to 1. Hence the real-valued function $v_i(p_j(t))$ is a continuous, strictly increasing bijection from $[0, 1]$ to itself. Its continuous inverse gives the *unique path parameter* at which person i attains any given level. Applying p_j then gives the unique indifferent path bundle, with a continuous dependence on the target level. This is the path-equivalence claim. □

A common calibration of either path is always possible. These two path-calibrated scales are coordinates for the full-class characterization, not scales claimed to satisfy EC individually.

Proposition O.2 (Reference-based welfare scales exist)**.** *Under Assumption 1, there exist welfare scales* $\mathbf{m}^1, \mathbf{m}^2$ *calibrated to* E_1, E_2*. For* $x \in E_1$ *and* $y \in E_2$*,*

$$m_1^1(x) \geq m_2^1(y) \iff x \succsim_2 y, \qquad m_1^2(x) \geq m_2^2(y) \iff x \succsim_1 y.$$

Proof of Proposition O.2. To calibrate a scale to path E_i, start with any continuous normalized ordinal representation m_i^i of the *path owner's* preferences. For the other person, find the unique point $\pi_i(x) \in E_i$ that she regards as equally good as her bundle x. Give her x the same number that the path owner assigns to that point:

$$m_{-i}^i(x) = m_i^i(\pi_i(x)).$$

If v_{-i} is any continuous normalized representation of $\succsim_{-i}$, then

$$m_{-i}^i = (m_i^i \circ p_i) \circ (v_{-i} \circ p_i)^{-1} \circ v_{-i}.$$

Each map in this construction is continuous and preserves the person's preference order. Thus m_{-i}^i is a continuous normalized ordinal representation of the other person's preferences. If the bundle x itself lies on E_i, its indifferent path point is x, and both people are then assigned the same label at x, which proves that a scale calibrated to the chosen path *exists*.

We must now determine what *cross-path comparisons* this calibration entails. If rural-path bundle x is given to either town, calibration makes the two labels at that physical bundle equal:

$$m_1^1(x) = m_2^1(x).$$

Because m_2^1 represents town 2's preferences, $m_1^1(x) \geq m_2^1(y)$ holds exactly when $x \succsim_2 y$; calibration to E_2 gives the symmetric comparison using town 1's preferences. Thus the two reference evaluations can disagree. □

Upper-anchor normalization. The full physical bundle 1 lies on both efficient reference paths; applying EC in both directions at $(1, 1)$ requires $\mathbf{e}_1(1) \sim \mathbf{e}_2(1)$. By Lemma O.1, their numerical labels coincide, $m_1(1) = m_2(1) \equiv M > 0$; along the diagonal bundles $t1$, each person's continuous representation reaches every label between 0 and M. Dividing both indices by the *same* M therefore normalizes the range to $[0, 1]$; relabeling the social ordering accordingly leaves strong Pareto, single-recipient anonymity, and any imposed full anonymity intact.

□

In the next lemma, a cross-person indifference is a pair $x \in E_1$, $y \in E_2$ with $m_1(x) = m_2(y)$. At such a tie, consensus rules out unanimous weak preference with at least one strict comparison. The converse also holds because the reference paths are continuously ordered. The restriction to reference bundles is essential; the lemma does not impose this condition on arbitrary physical bundles.

Lemma O.2. *Under Assumption 1,* $\succsim$ *satisfies EC if and only if at every cross-person indifference* (x, y)*, the two agents either disagree on the strict ranking of* x *and* y *or are both indifferent. Equivalently,*

$$x \succsim_1 y \iff y \succsim_2 x, \qquad x \succ_1 y \iff y \succ_2 x.$$

Proof. For necessity, suppose the authority ties the two recipients even though both towns weakly prefer one of the physical bundles and at least one town strictly prefers it. EC requires a *strict* social preference, contradicting the tie.

For sufficiency, suppose no forbidden social tie occurs. Take rural-path x and city-path y, both weakly ranked by the towns in favor of x. Let y' be the city-path bundle that the authority ties with x. If the city preferred y strictly to y', then both towns would weakly prefer x to y', one strictly, creating the prohibited tie. Hence $y' \succsim_2 y$. Along the city path, strong Pareto now gives $\mathbf{e}_1(x) \sim \mathbf{e}_2(y') \succsim \mathbf{e}_2(y)$. If either town strictly preferred x to y, social indifference would itself be forbidden, so the last ranking is strict; the reverse-agreement case follows by exchanging the towns.

□

O.2. SHARED-PATH COMPARISONS AND COMPENSATED SORTING

This section supplies the contour and finite-family results used to establish minimality in the main paper. Maintain strong monotonicity (SM), let $\mathcal{A}$ denote the common admissible paths from 0 to 1, and let $q_i^Q(x)$ be the unique point on Q indifferent to x for person i. Numerical labels m_i^Q use the same increasing calibration of Q for both people.

Every bundle can be reached by an admissible path. Thus the possible representatives of a welfare position are exactly its indifference contour.

Lemma O.3 (Admissible paths cover the bundle space)**.** *Let $X = [0,1]^L$, and suppose both preferences are continuous and satisfy (SM). For every $z \in X$, there exists $Q \in \mathcal{A}$ with $z \in Q$. Consequently, for every agent i and bundle x,*

$$\{q_i^Q(x) : Q \in \mathcal{A}\} = I_i(x) \equiv \{z : z \sim_i x\}.$$

Proof of Lemma O.3. Fix any bundle z other than 0 or 1. We construct a shared path through it by first raising resources from 0 to z, then from z to 1:

$$p_z(t) = \begin{cases} 2tz, & 0 \le t \le \frac{1}{2}, \\ z + (2t-1)(1-z), & \frac{1}{2} \le t \le 1. \end{cases}$$

Along each segment of the path, resources never decrease and at least one resource strictly increases. Both people therefore strictly prefer moving forward, by strong monotonicity; at $t = 1/2$ the path passes through the chosen z; endpoints $z = 0, 1$ can be handled by the diagonal $t1$. Hence any bundle z on person i's indifference curve through x is the point selected by some admissible shared path as the reference equivalent of x. Conversely, every selected equivalent must lie on that curve. This proves equality of the two sets in the lemma. □

The other person's lowest evaluation on that contour determines whether a comparison survives every path. The highest evaluation determines the reverse comparison.

Theorem O.1 (Robustness across reference paths is determined by indifference contours)**.** *Under the assumptions of Lemma O.3, fix $i \neq j$ and $x, y \in X$. Then*

$$m_i^Q(x) \ge m_j^Q(y) \text{ for every } Q \in \mathcal{A} \iff z \succsim_j y \text{ for every } z \in I_i(x). \tag{O.2.1}$$

The corresponding strict comparison holds for every $Q \in \mathcal{A}$ if and only if $z \succ_j y$ for every $z \in I_i(x)$.

Equivalently, for any continuous normalized representation $u_j : X \to [0,1]$, define

$$\underline{\gamma}_{ji}(x) = \min_{z \in I_i(x)} u_j(z), \qquad \overline{\gamma}_{ji}(x) = \max_{z \in I_i(x)} u_j(z).$$

Then the all-path robust comparisons are characterized by

$$\mathbf{e}_i(x) \succsim^{\mathcal{A}} \mathbf{e}_j(y) \iff \underline{\gamma}_{ji}(x) \ge u_j(y), \tag{O.2.2}$$

$$\mathbf{e}_j(y) \succsim^{\mathcal{A}} \mathbf{e}_i(x) \iff u_j(y) \ge \overline{\gamma}_{ji}(x). \tag{O.2.3}$$

Proof of Theorem O.1. For any fixed Q, calibration equates $m_i^Q(x)$ with $m_j^Q(q_i^Q(x))$. Since the latter scale represents j's own preferences, we have

$$m_i^Q(x) \ge m_j^Q(y) \iff q_i^Q(x) \succsim_j y,$$

and the comparison is strict exactly when the corresponding physical preference of j is strict. By the path-richness lemma, varying Q makes $q_i^Q(x)$ range over *every* bundle on i's indifference curve.

Therefore the welfare comparison holds under *all* path calibrations precisely when j weakly prefers every one of these equivalents to y. Continuity makes the contour closed, and X is compact, so u_j attains minimum and maximum values there; checking those extrema is equivalent to the all-bundle conditions and gives (O.2.2) and (O.2.3).

□

A reference family in the following result is nonempty. For a finite family $\mathcal{P} \subseteq \mathcal{A}$, define

$$\underline{\gamma}^{\mathcal{P}}_{ji}(x) = \min_{Q \in \mathcal{P}} u_j(q^Q_i(x)), \qquad \overline{\gamma}^{\mathcal{P}}_{ji}(x) = \max_{Q \in \mathcal{P}} u_j(q^Q_i(x)).$$

The normalization of u_j is immaterial. The family preserves the full weak comparison exactly when these extrema equal those over the entire contour.

Theorem O.2 (Finite sufficiency is equivalent to attaining both contour extremes)**.** *Under the assumptions of Lemma O.3, a finite family $\mathcal{P} \subseteq \mathcal{A}$ reproduces the all-path robust interpersonal preorder for every pair of single-recipient positions if and only if, for every $i \neq j$ and every $x \in X$,*

$$\underline{\gamma}^{\mathcal{P}}_{ji}(x) = \underline{\gamma}_{ji}(x), \qquad \overline{\gamma}^{\mathcal{P}}_{ji}(x) = \overline{\gamma}_{ji}(x). \tag{O.2.4}$$

The condition is invariant to the normalized utility representation used to express the extrema.

Proof of Theorem O.2. For a finite collection of paths, the all-path criterion has the same form, but its minima and maxima range only over the selected path representatives. If the selected paths attain the true lower and upper contour values, the finite and all-path tests have identical weak and strict thresholds against every opposing bundle. This proves sufficiency.

For necessity, suppose the selected family misses the *lowest* value that person j assigns to person i's contour. Choose a bundle y whose value for j lies strictly between this missed minimum and the lowest value seen on a selected path. Every chosen path then ranks i's position above j's, while a path through the true minimizer reverses that conclusion; missing the *maximum* similarly creates a false ranking in the opposite direction. Continuous, strongly monotone preferences attain every intermediate utility level, so these counterexample bundles exist. An increasing relabeling changes the numerical extrema but not whether the selected paths actually reach them. □

The following sorting lemma places the contour extrema on the two boundary paths. It is the technical step in the main proof of Theorem 2.

Lemma O.4 (Finite compensated sorting)**.** *Under continuity and (SM), if $x_r < 1$ and $x_s > 0$ for $r < s$, there is $y \triangleright_{rs} x$ with $y \sim x$ and either $y_r = 1$ or $y_s = 0$. Finitely many such transfers take every bundle to the unique point of G^- on its indifference contour; reversing the resource order gives the unique point on G^+.*

Proof. To construct one compensated transfer, hold all coordinates other than r and s fixed and move along the following L-shaped path in those two resource coordinates:

$$(x_r, 0) \longrightarrow (1, 0) \longrightarrow (1, x_s).$$

At the first endpoint the person has lost some of resource s without gaining r, so the bundle is strictly worse than x. At the last endpoint she has gained r without losing s, so it is strictly better. Along the L-shaped path, at least one resource increases and none falls; utility therefore rises strictly and continuously; exactly one point y is indifferent to x, with $y_r > x_r$ and $y_s < x_s$. It lies either on the horizontal segment, where $y_s = 0$, or on the vertical segment, where $y_r = 1$. Thus one compensated transfer preserves the person's welfare while filling an earlier resource or exhausting a later one.

This argument also covers boundary bundles: $x_r < 1$ and $x_s > 0$ alone ensure that the two endpoints lie strictly below and above x; to finish the sorting, take the first resource coordinate r still

below one. If all later coordinates are zero, the bundle is already on G^-; otherwise compensate a rise in r by reducing some later positive coordinate. Each transfer either fills r permanently or empties a later coordinate, so at most $L - r + 1$ transfers are needed before moving on; across resources there are at most $\sum_{r=1}^{L-1}(L - r + 1) = (L - 1)(L + 2)/2$ steps. The final bundle lies on G^-, and strict ordering of that path gives a unique intersection with the initial indifference contour; reversing the resource order yields G^+ with the same bound.

□

O.3. POPULATION COMPATIBILITY AND THE MATCHING CONSTRUCTION

The population result is needed because independently permissible bilateral comparisons need not admit one common scale. Let $G = (N, \mathcal{E})$ be a finite connected graph of declared bilateral comparison problems. Each edge has regular paths with common physical endpoints 0, 1. A globally compatible scale represents every person's preferences continuously, has common endpoint labels, and respects every edge's EC restriction; $\mathcal{M}^{EC}$ is the class of these scales.

Let $\succeq^0$ collect the direct bilateral and own-preference comparisons. A strict cycle is a finite chain returning to its starting welfare position with at least one strict link. Write $\succeq^*$ for the transitive closure. The following result establishes existence and identifies the comparisons forced by every compatible scale. Section O.6 below provides an economic example of failure.

Theorem O.3 (Population consistency and the EC-robust interpersonal preorder)**.** *For any finite connected comparison graph G as above, the class $\mathcal{M}^{EC}$ is nonempty if and only if $\succeq^0$ has no strict cycle. When this condition holds, let $\succeq^*$ denote the transitive closure of $\succeq^0$. Then*

$$(i, x) \succeq^{EC} (j, y) \quad \Longleftrightarrow \quad (i, x) \succeq^* (j, y), \tag{O.3.1}$$

where the left side means $m_i(x) \geq m_j(y)$ for every $\mathbf{m} \in \mathcal{M}^{EC}$. The comparison is strict under every compatible scale if and only if $(i, x) \succeq^* (j, y)$ and $(j, y) \not\succeq^* (i, x)$.

Proof of Theorem O.3. Each bilateral EC band can be expressed as a numerical *threshold*. For a given own-preference level of town i, ask how high town j's own-preference level can be while EC still guarantees that it is no better. We propagate those thresholds along chains, construct a continuous common scale from the strongest reachable comparisons, and show that an unforced comparison can be reversed; the case $n = 1$ is immediate, so let $n \geq 2$. Choose continuous normalized own-preference representations v_i, and write (i, t) for an i-position at level t. On each oriented edge $i \to j$, let $\ell_{ij}(t)$ and $u_{ij}(t)$ be the EC band's *lower and upper endpoints* in v_j-levels; the bilateral theorem makes both maps increasing homeomorphisms and gives

$$u_{ij} = \ell_{ji}^{-1}, \tag{A.P1}$$

and

$$(i, t) \succeq^0 (j, s) \iff s \leq \ell_{ij}(t). \tag{A.P2}$$

Thus the direct comparison $(i, t) \succeq^0 (j, s)$ holds exactly when $s \leq \ell_{ij}(t)$. At a nondegenerate band's boundary, the required comparison is strict; at a pinned band it is a tie; the same person's levels are ordered by her own preferences. Any common numerical scale must respect all these direct links. A closed chain containing a strict link would then require its starting position to rank strictly above itself, proving why the no-strict-cycle condition is necessary.

Now assume no strict cycle and construct a scale. If the sequence of compared towns is $P = (i_0, \ldots, i_K)$, compose the lower thresholds along its edges: $\ell_P = \ell_{i_{K-1} i_K} \circ \cdots \circ \ell_{i_0 i_1}$. Starting from person i_0's level t, the number $\ell_P(t)$ is the highest terminal level that the *successive direct comparisons* guarantee is no better. Indeed, a closed detour cannot improve the guaranteed level without creating a strict cycle:

$$\ell_C(r) \leq r \tag{A.P3}$$

for every r. If $\ell_C(r) > r$, the closed chain reaches a strictly higher level of the starting person, contradicting the absence of strict cycles. If $\ell_C(r) = r$, every link of that closed chain must be a tie, since a strict link would again create a strict cycle. Thus $\ell_C(r) \leq r$; monotonicity of subsequent threshold maps means that deleting the detour cannot reduce the final guaranteed level. Removing a repeated-town segment therefore cannot weaken the terminal lower bound; only *simple paths*, which visit each town at most once, need to be checked, and there are finitely many of them.

Let $\mathcal{P}^s_{ij}$ be the finite set of simple paths from i to j and define

$$\Lambda_{ij}(t) = \max_{P \in \mathcal{P}^s_{ij}} \ell_P(t), \qquad \Lambda_{ii}(t) = t. \tag{A.P4}$$

The preceding observations imply

$$(i,t) \succeq^* (j,s) \iff s \leq \Lambda_{ij}(t). \tag{A.P5}$$

For each starting position (i,t), collect its largest guaranteed threshold at *every* town into the vector $V_i(t) = (\Lambda_{i1}(t), \ldots, \Lambda_{in}(t))$. If a comparison from (i,t) to (j,s) follows a chain, then anything reached from (j,s) can also be reached from (i,t), and hence $V_i(t)$ is at least $V_j(s)$ in every coordinate. Conversely, the coordinate for town j recovers the original comparison because $\Lambda_{jj}(s) = s$, and we therefore obtain an exact numerical description of the transitive closure:

$$(i,t) \succeq^* (j,s) \iff V_i(t) \geq V_j(s) \quad \text{coordinatewise;} \tag{A.P6}$$

the converse follows from coordinate j, since $\Lambda_{jj}(s) = s$.

Each coordinate Λ_{ik} is the maximum of *finitely many* continuous, strictly increasing path-threshold functions. It is therefore continuous, strictly increasing, and normalized at the endpoints; the vector $V_i(t)$ records all the direct comparisons implied by person i's level t. Give each coordinate a positive weight w_k, with the weights summing to one, and define the *candidate* common scale

$$\widehat{m}_i(t) = \sum_{k=1}^{n} w_k \Lambda_{ik}(t). \tag{A.P7}$$

The formula is continuous and strictly represents each person's own ranking. To verify EC, examine a bilateral band boundary $s = \ell_{ij}(t)$. The threshold vector of (i,t) weakly dominates that of (j,s). If the band has positive width, equality of the vectors would imply a reverse chained comparison, which together with the strict boundary comparison would form a forbidden strict cycle. Thus at least one coordinate is strictly greater. Positive weights preserve that strict inequality; applying the same argument to the reverse directed edge and using (A.P1) yields the two inequalities

$$\widehat{m}_j(\ell_{ij}(t)) < \widehat{m}_i(t) < \widehat{m}_j(u_{ij}(t))$$

inside every nondegenerate band, with equalities only at pinned points. After composing these own-level labels with v_i, we obtain continuous interpersonal scales whose social equivalents lie inside every pair's EC band; one population scale therefore exists, proving sufficiency.

We still need to prove that the transitive closure contains *every* comparison that all compatible scales share. Suppose $(i,t) \not\succeq^* (j,s)$. Then the target s exceeds the greatest level at j reachable from (i,t) by $\delta = s - \Lambda_{ij}(t) > 0$. In the scale just constructed, put almost all weight on the coordinate for town j, and give the remaining coordinates a small common positive weight ε:

$$0 < \varepsilon < \frac{\delta}{(n-1)(1+\delta)}.$$

All threshold coordinates lie in $[0,1]$. Choosing ε small relative to the positive gap δ therefore makes this *compatible* scale rank (j,s) strictly above (i,t). An unchained comparison cannot be universal; for the strict part, suppose a forward chained comparison exists but no reverse chain does. If every link were reversible, the whole chain could be reversed, and hence some link must be strict; every compatible scale then preserves that strict ranking. This proves the exact weak and strict characterizations.

The vectors also represent the closure without choosing weights. On $D = \bigcup_i V_i([0,1])$, any compatible scale factors as a continuous product-strict function Ψ: equal vectors give two-way chained comparisons and equal labels, while the restrictions to the finitely many compact score curves are continuous and agree on intersections. Conversely, such a conversion preserves all required weak and strict comparisons.

An increasing relabeling η_i of each person's own utility coordinate conjugates the edge threshold to $\eta_j \circ \ell_{ij} \circ \eta_i^{-1}$, and likewise conjugates every path threshold. Thus neither strict cycles nor chained comparisons depend on arbitrary ordinal labels. On disconnected graphs the construction applies to each component separately; independently relabeling the components leaves only the common endpoint comparisons robust across components. □

A spanning tree avoids contradictions from multiple routes between recipients. It is minimal in edge count among connected comparison graphs; this is a statement about comparison design, not about a uniquely appropriate set of bilateral institutions.

Corollary O.1 (A spanning tree is a minimal connected comparison design)**.** *For $n \geq 2$, any connected bilateral comparison network has at least $n-1$ edges. Every spanning tree attains this bound and is globally EC-compatible. Its EC-robust comparisons between any two agents are exactly those implied by chaining the direct comparisons along their unique simple path.*

Proof of Corollary O.1. A connected graph needs at least $n-1$ edges, and a *spanning tree* has exactly that many. Choose a root town, calibrate a scale for it, and extend the scale to each neighboring town through the tree's chosen pairwise equivalence. Every new town has only one already calibrated neighbor, so no second route can demand conflicting labels. The global theorem therefore applies, and the robust comparisons are those obtained by chaining. Adding edges enlarges the required direct relation and its closure; an edge is incompatible precisely if it creates a strict cycle, redundant if it adds no comparison, and informative otherwise.

□

The matching converse combines finitely many witness scales into one scale and separates a finite set of welfare vectors by a strictly increasing function. The next two lemmas justify those steps.

Lemma O.5 (Common increasing transformations preserve EC compatibility)**.** *Let $\mathbf{m}^1, \ldots, \mathbf{m}^K \in \mathcal{M}^{EC}$, and let $\Phi : [0,1]^K \to \mathbb{R}$ be continuous and strictly increasing in the product order. Define*

$$\widetilde{m}_i(x) = \Phi\big(m_i^1(x), \ldots, m_i^K(x)\big).$$

After one common positive affine normalization, the population scale $\widetilde{\mathbf{m}} = (\widetilde{m}_i)_{i \in N}$ belongs to $\mathcal{M}^{EC}$.

Proof. Each component scale correctly ranks a person's own bundles. A common continuous conversion strictly increasing in all component coordinates preserves those rankings and every required strict EC comparison. All persons share the endpoint vectors 0^K and 1^K, so one common positive affine normalization restores the endpoint labels, and thus the combined scale remains EC-compatible. We use this closure property below to turn several separate counterexamples into *one* scale that defeats a policy ranking. □

Lemma O.6 (Finite monotone separation)**.** *Let $K \geq 1$ and $F \neq \varnothing$ be a finite subset of $[0,1]^K$, and let $>^*$ be a strict total order on F that extends strict product dominance. There exists a continuous function $\Phi : [0,1]^K \to \mathbb{R}$ that is strictly increasing in the product order and represents $>^*$ on F.*

Proof. The next elementary construction will let us implement a desired ranking of *finitely many* welfare vectors without breaking coordinatewise dominance. Suppose we have a finite set F whose vectors must respect strict product dominance, together with a strict total ranking extending those required comparisons; assign the vectors integer ranks $r(z)$. We penalize any attempt to give a vector a higher value merely because of a coordinate not present in the proposed lower vector; choose M large enough that such a penalty outweighs every integer-rank difference. Specifically, let

$$\delta = \min_{\substack{p,q\in F \\ q\not\leq p}} \sum_{\ell=1}^{K} (q_\ell - p_\ell)_+ > 0$$

when such pairs exist and set $\delta = 1$ otherwise. For $M > |F|/\delta$, define

$$g(z) = \max_{q\in F} \left\{ r(q) - M \sum_{\ell=1}^{K} (q_\ell - z_\ell)_+ \right\}.$$

The function g is continuous and nondecreasing in each coordinate. At any original vector $p \in F$, the term $q = p$ achieves $r(p)$; every $q \leq p$ has no higher rank by the assumed extension of product dominance; any $q \not\leq p$ pays a sufficiently large penalty. Hence $g(p) = r(p)$ exactly. Because the prescribed ranks are distinct integers, choose $0 < \varepsilon < 1/K$; adding $\varepsilon \sum_\ell z_\ell$ then makes $\Phi(z) = g(z) + \varepsilon \sum_\ell z_\ell$ strictly increasing under every genuine coordinatewise improvement while preserving the prescribed ranking of the finite set.

□

O.4. COMMON REFERENCES AND THE FACILITY APPLICATION

Common physical references allow every person's welfare position to be represented by the same two ordinal coordinates. Maintain the common-reference condition (CR) and the notation S_i, D_N from the main paper.

Theorem O.4 (Common reference paths reduce population comparison to two scores)**.** *Under* (CR)*, a normalized population welfare scale* $\mathbf{m} = (m_i)_{i\in N}$ *satisfies every bilateral EC restriction if and only if there is one continuous function* $\Psi : D_N \to [0,1]$ *that is strictly increasing in the product order, has* $\Psi(0,0) = 0$ *and* $\Psi(1,1) = 1$*, and satisfies*

$$m_i(x) = \Psi\big(S_i(x)\big) \qquad \text{for every } i \in N,\ x \in [0,1]^L. \tag{O.4.1}$$

Hence the class $\mathcal{M}^{EC}$ *is nonempty.*

Proof of Theorem O.4. Under EC, the bilateral characterization gives an increasing two-score conversion on each person's compact score image D_i. Continuity of the utility and path-location maps makes each restriction continuous; they agree on intersections and therefore paste to one continuous Ψ on the finite union D_N. Conversely, any continuous normalized product-strict conversion satisfies all pairwise EC bands, and $\Psi(s,t) = (s+t)/2$ establishes existence without identifying cardinal weights. □

Population OCA makes those common references arise from efficient sharing. For the additional population-wide minimality conclusion, impose the stronger same-bundle condition

$$\begin{aligned} y \triangleright_{rs} x, \quad y \sim_i x &\Longrightarrow x \succ_j y, \\ y \triangleright_{rs} x, \quad y \sim_j x &\Longrightarrow y \succ_i x \end{aligned} \qquad (i < j). \tag{P-OSC}$$

A common reference family is population-sufficient when it preserves every pair's weak all-path comparisons. This is a requirement within one resource pool.

Proposition O.3 (Ordinal comparative advantage generates common reference paths)**.** *Under* (P-OCA)*, for every $i < j$ the efficient projections of the full bilateral sharing problem are $E_i^{ij} = E^- = G^-$ and $E_j^{ij} = E^+ = G^+$, with G^-, G^+ defined in* (4.3)*. Hence* (CR) *holds for every finite L, all bilateral EC restrictions are globally compatible, and Theorem O.4 together with the coordinatewise matching criterion* (5.3) *apply.*

Corollary O.2 (Two common reference paths are necessary and sufficient population-wide)**.** *Suppose $n \geq 2$ and maintain* (P-OCA) *and* (P-OSC)*. The pair $\{G^-, G^+\}$ is sufficient for the population, no single common path is, and any sufficient two-path family equals $\{G^-, G^+\}$ up to relabeling. Hence these paths are the unique minimum common reference system recovering all pairwise all-path robust comparisons.*

Proofs of Proposition O.3 and Corollary O.2. Apply the bilateral sorting theorem to every ordered pair $i < j$. P-OCA implies $E_i^{ij} = G^-$ and $E_j^{ij} = G^+$ for all such pairs, so the physical references coincide and the population comparison theorem supplies a common scale. If P-OSC also holds, every pair has the same uniquely minimal efficient-path family. That pair is therefore uniquely minimal for the entire population. □

The two-ranking assignment theorem in the main paper immediately gives a test for a best assignment. Both treatment-state requirements must be met; a high treated score alone is insufficient.

Corollary O.3 (When consensus identifies a best fixed-K assignment)**.** *Under the assumptions of Theorem 4, p^A is weakly preferred to every K-recipient policy by all anonymous EC social preferences if and only if A contains K highest h_i's and K lowest ℓ_i's, allowing ties. When all these scores are distinct, a universally best assignment exists exactly when the two K-recipient sets coincide, and it is then strictly preferred to every other assignment.*

Proof of Corollary O.3. Choosing the K highest treated scores makes every treated rank as high as possible. Leaving the $n - K$ highest untreated scores unselected makes every untreated rank as high as possible. If one recipient set achieves both, the two-order theorem makes it universally best. Conversely, if a candidate misses either set, another assignment improves the corresponding ordered list and prevents it from dominating every rival. With distinct scores the two optimal sets, if they coincide, are unique and every change lowers a rank strictly. □

O.4.1. Verification of the three-town example

In physical units the example has $a = 1/6$, $R_a = 5/6$, and $u_i(s, z) = g_i(s) + z$, where

$$g_1(s) = \frac{99s - 11s^2}{40}, \quad g_2(s) = \frac{63s + 11s^2}{40}, \quad g_3(s) = \frac{71s - 11s^2}{40}.$$

All derivatives are positive on $[0, 1]$. The differences between marginal valuations at complementary bundles are

$$g_1'(s) - g_2'(R_a - s) = \frac{18 - 11R_a}{20} > 0, \qquad g_2'(s) - g_3'(R_a - s) = \frac{-4 + 11R_a}{20} > 0,$$

and $g_1'(s) - g_3'(R_a - s) = (14 + 11R_a - 22s)/20 > 0$ for $s \in [0, R_a]$. Thus the compensated complementary comparison gives population OCA. Same-bundle sorting fails because $g_2'(s) - g_3'(s) = (-4 + 22s)/20$ changes sign.

The untreated answers are

$$z_1 = \frac{583}{1440}, \qquad z_2 = \frac{389}{1440}, \qquad z_3 = \frac{415}{1440}.$$

Solving $g_i(q_i) = g_i(a) + 1$ gives approximately $0.609, 0.717, 0.833$, with $q_3 = 5/6$ exactly. These answers all lie on the stated reference paths. Their strict rankings give the one-facility incomparability and two-facility optimum directly from Theorem 4. The calculation holds at the stated reference endowment; it does not assert invariance to changes in the sharing institution.

O.4.2. Why stronger sorting reverses the two state rankings

Maintain the facility environment of Section 5 in the main paper and add (P-OSC). We show directly that the treated scores h_i strictly increase with the town index while the untreated scores ℓ_i strictly decrease. The resulting top-K conclusion uses no further allocation theorem.

Fix $i < j$, write $L_b = (b, 0)$, $H_b = (b, 1)$ with $0 < b < 1$, and let $q_i^-(H_b)$ be town i's indifferent representative of H_b on E^-. By Theorem 2, on this i-contour town j's evaluation is uniquely minimized at $q_i^-(H_b)$ and maximized at $H_b \in E^+$. These bundles are distinct, so $H_b \succ_j q_i^-(H_b)$. Town j's own representative of H_b therefore lies farther along E^-. The common parametrization gives $h_j > h_i$.

For the untreated state, let $q_j^+(L_b)$ be town j's representative on E^+. On the j-contour through L_b, town i's evaluation is uniquely maximized at $L_b \in E^-$ and minimized at $q_j^+(L_b)$. Thus $L_b \succ_i q_j^+(L_b)$, and town i's representative lies farther along E^+, giving $\ell_i > \ell_j$. Both arguments use interior state bundles on opposite boundary paths, so neither relies on an equality at the empty or full bundle.

If an assignment treats i and leaves $j > i$ untreated, transferring the facility to j strictly improves both matched positions: $h_j > h_i$ and $\ell_i > \ell_j$. Repeating such transfers reaches exactly the K highest indices. Every other assignment is therefore strictly worse under every anonymous EC evaluation. This conclusion also holds for any $0 < b < 1$; the baseline accounting in the main application imposes the narrower bound $b \le 1/2$.

O.5. REGULAR REFERENCE PATHS AND ORDINAL SORTING

The reference-path construction requires efficient personal projections that run continuously from the empty bundle to the full endowment and are ordered by each person's own preference. Comparative advantage gives a familiar economic sufficient condition in the two-resource example.

Proposition O.4 (Comparative advantage generates regular reference paths)**.** *Suppose*

$$\mathcal{X} = \{(x_1, x_2) \in [0,1]^2 \times [0,1]^2 : x_1 + x_2 \le (1,1)\},$$

and each preference admits a C^1 representation u_i with strictly positive partial derivatives. If, after relabeling agents if necessary,

$$\frac{u_{1s}(x)}{u_{1z}(x)} > \frac{u_{2s}(1-x)}{u_{2z}(1-x)} \qquad \text{for every } x = (s,z) \text{ with } s < 1,\ z > 0, \tag{CA}$$

then the regular-reference-path assumption in the main paper holds. In particular,

$$E_1 = \{(s,0) : s \in [0,1]\} \cup \{(1,z) : z \in [0,1]\},$$
$$E_2 = \{(0,z) : z \in [0,1]\} \cup \{(s,1) : s \in [0,1]\}.$$

Proof of Proposition O.4. Positive marginal utilities make wasting any available resource inefficient, so a candidate efficient division is $(x, 1-x)$, with $x = (s,z)$. We first show that a bundle giving person 1 *some specialist care before all general care is allocated to her* cannot be efficient. If $s < 1, z > 0$, the inequality (CA) places a number q strictly between the people's marginal rates of substitution. Transfer a small amount ε of general care to person 1 and $q\varepsilon$ of specialist services to person 2. Because person 1 has room for more care and positive specialist service to give up, the transfer is feasible. The directional derivative of agent 1's utility is $u_{1s}(x) - qu_{1z}(x) > 0$, while agent 2's is $-u_{2s}(1-x) + qu_{2z}(1-x) > 0$. Both therefore gain from a sufficiently small feasible transfer. The original interior division was therefore Pareto-dominated, and thus person 1 can receive efficient bundles only on the capacity-first boundary $z = 0$ or $s = 1$. We must still show that every bundle on those boundaries is efficient.

Conversely, suppose a division with person 1 on the lower edge is Pareto-dominated. Maximizing the sum of both continuous utilities over the nonempty compact set of weak improvements produces

a Pareto-efficient division that also dominates the original. The preceding argument places the first recipient of this maximizer on the lower or right boundary. No distinct division there can dominate the original: a move toward more general care on the lower edge takes resources from person 2, the reverse move hurts person 1, and moving to the right edge also takes strictly valued resources from person 2. The upper-edge case is symmetric.

Taking complements gives person 2's claimed projection. Both paths are continuous from 0 to 1, and each forward move raises at least one resource without lowering another. Positive marginal utilities make both people strictly prefer every forward move, proving regularity. □

A different sufficient condition uses strict concavity and complementarity rather than a uniform ranking of the two people's marginal rates of substitution. It establishes regularity, not the stronger sorting assumptions used for minimal reference sufficiency.

Proposition O.5 (Concave supermodular preferences generate regular reference paths)**.** *Suppose the feasible set is the full L-good resource problem*

$$\mathcal{X} = \{(x_1, x_2) \in [0,1]^L \times [0,1]^L : x_1 + x_2 \leq 1\}, \qquad L \geq 2.$$

If each preference admits a C^2, strictly increasing, strictly concave representation u_i with

$$\frac{\partial^2 u_i(x)}{\partial x_\ell \, \partial x_k} \geq 0 \qquad \text{for all } \ell \neq k,$$

then the regular-reference-path assumption in the main paper holds.

Proof of Proposition O.5. We support efficient allocations by welfare weights, use supermodular comparative statics to order the supported bundles, and then verify continuity and common ranking of the resulting projected paths. Strict monotonicity implies that every Pareto-efficient allocation exhausts all resources, so write a full allocation as $(x, 1-x)$ with $x \in [0,1]^L$. For each normalized Pareto weight $t \in [0,1]$, consider

$$x(t) \in \arg\max_{x \in [0,1]^L} \{t\, u_1(x) + (1-t) u_2(1-x)\}.$$

The maximizer is unique. For $t \in (0,1)$, this follows from strict concavity; at $t = 0$ and $t = 1$, strict monotonicity gives $x(0) = 0$ and $x(1) = 1$.

The objective is supermodular in x: the cross partials of both $u_1(x)$ and $u_2(1-x)$ are weakly nonnegative. It also has increasing differences in (x, t). Indeed, for $t' > t$, the difference between the two objectives is

$$(t' - t)\big[u_1(x) - u_2(1-x)\big],$$

and the bracketed term is strictly increasing in the product order because $u_1(x)$ rises with x while $u_2(1-x)$ falls. The monotone-comparative-statics theorem for supermodular optimization therefore gives

$$t' > t \quad \Longrightarrow \quad x(t') \geq x(t)$$

coordinatewise (Topkis, 1998). If the two maximizers are distinct, at least one coordinate inequality is strict.

Every Pareto-efficient allocation is supported by some nonnegative welfare weights. Concavity of the two utility representations and convexity of the feasible set make the comprehensive utility possibility set convex, so a supporting hyperplane at any Pareto-efficient utility pair supplies such weights. After normalization, every efficient allocation is therefore $(x(t), 1-x(t))$ for some $t \in [0,1]$. Hence

$$E_1 = \{x(t) : t \in [0,1]\}, \qquad E_2 = \{1 - x(t) : t \in [0,1]\}.$$

The maximum theorem and uniqueness make $x(t)$ continuous. Thus E_1 is a compact connected coordinatewise chain from 0 to 1. Any two distinct points on this chain are coordinatewise ordered with at least one strict coordinate, so strict monotonicity of each u_i makes both agents rank them in the same strict order. The restriction of u_1 to E_1 is therefore continuous and one-to-one, with range from $u_1(0)$ to $u_1(1)$; after normalization its inverse gives a homeomorphism from $[0,1]$ onto E_1. The same argument, reversing the Pareto-weight order, applies to E_2. Thus the regular-reference-path assumption in the main paper follows. □

For the unique-minimal-reference result, regularity alone is insufficient. The paper distinguishes comparative advantage in efficient resource division (OCA) from same-bundle single crossing (OSC). The next result gives differentiable conditions for these two separate restrictions; their conclusions use the main paper's already established sorting theorem.

Proposition O.6 (Differentiable sufficient conditions for the ordinal sorting assumptions)**.** *Let $L \geq 2$ and suppose u_1, u_2 are C^1, strictly increasing, and have positive partial derivatives on $[0,1]^L$. After ordering resources, suppose for every $r < s$,*

$$\frac{u_{1r}(x)}{u_{1s}(x)} > \frac{u_{2r}(1-x)}{u_{2s}(1-x)} \qquad \text{whenever } x_r < 1,\ x_s > 0, \tag{HD-CA}$$

and

$$\frac{u_{1r}(x)}{u_{1s}(x)} > \frac{u_{2r}(x)}{u_{2s}(x)} \qquad x \in [0,1]^L. \tag{HD-SC}$$

Then HD-CA *implies the main paper's ordinal comparative-advantage condition, and* HD-SC *implies its ordinal same-bundle single-crossing condition. Consequently the opposite greedy paths are the efficient projections and, under both inequalities, are the unique minimum-cardinality sufficient pair for all-path robust comparison.*

Proof. Fix $r < s$ and bundles $y \triangleright_{rs} x$ with $y \sim_1 x$, holding all other coordinates fixed. For every $q \in [x_r, y_r]$, strict increase in coordinate s places the utility level $u_1(x) = u_1(y)$ strictly between $u_1(q, y_s)$ and $u_1(q, x_s)$, except at the corresponding endpoints. Hence there is a unique $\phi(q) \in [y_s, x_s]$ with $u_1(q, \phi(q)) = u_1(x)$. The implicit-function theorem and $u_{1s} > 0$ make this entire indifference segment a C^1 graph $z_s = \phi(z_r)$ with

$$\phi' = -\frac{u_{1r}}{u_{1s}}.$$

Along this segment,

$$\frac{d}{dz_r} u_2(1-z) = u_{2s}(1-z)\left[\frac{u_{1r}(z)}{u_{1s}(z)} - \frac{u_{2r}(1-z)}{u_{2s}(1-z)}\right] > 0$$

under (HD-CA). Hence $1 - y \succ_2 1 - x$, which is ordinal comparative advantage. Under (HD-SC),

$$\frac{d}{dz_r} u_2(z) = u_{2s}(z)\left[\frac{u_{2r}(z)}{u_{2s}(z)} - \frac{u_{1r}(z)}{u_{1s}(z)}\right] < 0,$$

so $x \succ_2 y$. Repeating the calculation along a u_2-indifference segment gives $y \succ_1 x$ whenever $y \sim_2 x$, which is ordinal same-bundle single crossing. Lemma O.4 and Theorem 2 in the main paper then give the efficient projections, contour extrema, and minimality. □

O.6. A STRICT CYCLE OF BILATERAL EC COMPARISONS

Three independently acceptable bilateral reference problems need not admit one common population welfare scale. In this example, the preferences are positive and linear; the cycle comes from restricting which resource divisions are available to each compared pair.

Proposition O.7 (Restricted sharing can create global EC cycles)**.** *There are three positive linear preferences on* $[0,1]^3$ *and three compact bilateral reference problems with free disposal, each containing all single-recipient allocations and satisfying regularity, for which the direct EC comparisons contain a strict cycle.*

Proof. Consider three resources and town preferences $u_i(x) = a_i \cdot x$, where $a_1 = (5,2,1)$, $a_2 = (1,5,2)$, and $a_3 = (2,1,5)$; for any resource order σ, let $p_\sigma(t)$ allocate resources in that order as t increases. The pair $(1,2)$ uses $(2,1,3)$, pair $(2,3)$ uses $(3,2,1)$, and pair $(3,1)$ uses $(1,3,2)$. Each pair is restricted to divisions along its designated route, with free disposal:

$$\mathcal{X}^{ij} = \{(x_i, x_j) : 0 \le x_i \le p_\sigma(t),\ 0 \le x_j \le 1 - p_\sigma(t) \text{ for some } t \in [0,1]\}.$$

Each pair's feasible set is compact, allows disposal, and includes all one-recipient allocations by choosing $t = 0$ or 1. We can put the three problems in one feasible three-person economy, giving the outsider zero during each bilateral comparison. Since everyone strictly values more of every resource, discarding resources is inefficient; for a full division $(p_\sigma(t), 1 - p_\sigma(t))$, increasing t helps one person and hurts the other. These allocations are efficient, and their personal projections are regular paths both people rank monotonically.

Give each town the same bundle $z = (1/4, 1/4, 1/4)$, worth 2 in *each town's own ordinal utility representation*. Write e_r for one unit of resource r and nothing else. For pair $(1,2)$, town 1 regards e_2 as equal to z, and town 2 regards e_3 as equal to z. Both prefer e_2 to e_3, so EC ranks town 1 at z strictly above town 2 at z. Repeating the construction with e_3, e_1 for pair $(2,3)$ and e_1, e_2 for pair $(3,1)$ gives

$$(1,z) \succ^0 (2,z) \succ^0 (3,z) \succ^0 (1,z).$$

Every edge separately permits an EC scale. One population scale, however, would require $m_1(z) > m_2(z) > m_3(z) > m_1(z)$, which no real numbers satisfy. The error would be to treat separately permitted bilateral ties as if they automatically defined one population ranking. □

O.7. REFERENCE CHOICE AND THE LIMITS OF POLICY IDENTIFICATION

This section verifies the two qualifications used in Sections 4 and 5 of the main paper. First, efficient references can settle a choice that a different shared path reverses; a concavity result states the scope of that example. Second, changing the hypothetical specialist endowment can remove the assignment rankings even when preferences satisfy the stronger sorting conditions. All comparisons retain the same personal preferences and actual policy bundles.

O.7.1. A strict policy reversal across reference choices

The next example keeps preferences and physical outcomes fixed while changing the shared path used for interpersonal calibration. Efficient references settle the facility choice, but another path reverses it. The preferences are smooth and strongly monotone; they need not be concave.

Example O.1 (Efficient references can settle a choice left open by all-path robustness)**.** *Two towns have actual bundles* $L = (1/4, 0)$ *and* $H = (1/4, 1)$*; let* $p^1 = (H, L)$ *and* $p^2 = (L, H)$*. There are smooth, strongly monotone preferences such that, at every reference endowment* $(R, 1)$ *with* $41/50 \le R \le 1$*, every anonymous EC social preference strictly ranks* p^2 *above* p^1*, while calibration on another admissible shared path strictly ranks* p^1 *above* p^2 *under every anonymous, strongly Pareto evaluation. Neither policy comparison is therefore robust across all shared paths.*

The mechanism is the two welfare-level comparisons from the introduction. In the construction, the untreated bundles have specialist-service equivalents $z_1 \simeq 0.787 > z_2 \simeq 0.719$, while the treated bundles have general-care equivalents $q_2 \simeq 0.642 > q_1 \simeq 0.623$. The efficient paths therefore rank town 2 higher when treated and town 1 higher when untreated. Another shared path reverses both

comparisons. The following verification constructs the preferences and that path explicitly, verifies the inequalities uniformly over the stated budgets, and establishes their stability under small C^1 perturbations within the constructed quasilinear class. The extra decision therefore comes from accepting the efficient-reference restriction, not from recovering a previously hidden cardinal welfare gain. This is a policy-level illustration of the familiar tradeoff between reference robustness and decisiveness (Decancq and Nys, 2021), with the reference family selected by the resource-allocation problem.

The example does not contradict Theorem 2 in the main paper. Without the additional sorting conditions, EC can justify comparisons that do not survive recalibration on every shared path. Nor does the example show that efficient references are ethically obligatory. Within a concave quasilinear two-resource class, the comparative-advantage condition used to obtain these boundary paths already implies the stronger same-bundle sorting condition; Proposition O.8 proves this boundary. The construction here deliberately allows nonconcave preferences. An application of this example therefore needs a reason not to impose globally diminishing marginal valuations; smoothness alone does not provide that justification.

Verification of Example O.1. Work in physical units on $[0, R] \times [0, 1]$, and set

$$u_i(s, z) = g_i(s) + z, \qquad g_1(s) = \frac{11}{5}s + \frac{1}{2}s^2 + \frac{2}{9}\big[1 - (1 - s)^9\big], \qquad g_2(s) = 3s - \frac{1}{2}s^2.$$

Both preferences are smooth and strongly monotone on the relevant rectangles. The derivative comparison

$$g_1'(s) - g_2'(R - s) = R - \frac{4}{5} + 2(1 - s)^8 \geq \frac{1}{50} \quad (0 \leq s \leq R,\ 41/50 \leq R \leq 1)$$

establishes OCA after normalization. Indeed, if a rise in town 1's general care from s to t is offset by a specialist reduction $g_1(t) - g_1(s)$, town 2's gain at the complementary allocation is

$$\int_s^t \big[g_1'(v) - g_2'(R - v)\big]\, dv > 0.$$

The OCA part of Theorem 2 therefore identifies the efficient projections as the capacity-first and specialization-first boundary paths. No OSC premise is used in this step. Strict monotonicity makes the paths regular.

Write $a = 1/4$. The untreated equivalents on the specialization-first path satisfy

$$z_1 = g_1(a) = \frac{4640657}{5898240} > \frac{23}{32} = g_2(a) = z_2, \qquad 0 < z_2 < z_1 < 1.$$

The treated equivalents on the capacity-first path solve $g_i(q_i) = g_i(a) + 1$. Both lie below $41/50$, and

$$q_1 < \frac{63}{100} < q_2, \qquad q_1 \simeq 0.622977, \quad q_2 \simeq 0.641505.$$

For an exact check, substitution gives $g_1(63/100) - g_1(a) - 1 > 1/100$, while $g_2(63/100) - g_2(a) - 1 = -17/625 < 0$; evaluation at $41/50$ makes both residuals positive. The EC score criterion thus ranks town 1 higher when untreated and town 2 higher when treated. Anonymity and strong Pareto imply $p^2 \succ p^1$ at every stated R.

To reverse both comparisons using one admissible shared path, put $D = g_1 - g_2$. Direct rational evaluation gives

$$D(1/5) > D(1/4) > D(3/10).$$

Define

$$r_L = \big(1/5,\ g_1(1/4) - g_1(1/5)\big), \qquad r_H = \big(3/10,\ 1 + g_2(1/4) - g_2(3/10)\big).$$

Their specialist coordinates are approximately 0.134391 and 0.863750. Join $0, r_L, r_H, (R, 1)$ by straight segments to form Q_R. Both physical coordinates increase strictly along every segment, so both towns rank the path strictly from worst to best. It is an admissible shared reference.

Town 1 is indifferent between r_L and L, whereas

$$u_2(r_L) - u_2(L) = D(1/4) - D(1/5) < 0.$$

Town 2's equivalent of L is therefore farther along Q_R than town 1's, giving $m_2^{Q_R}(L) > m_1^{Q_R}(L)$. Town 2 is indifferent between r_H and H, whereas

$$u_1(r_H) - u_1(H) = D(3/10) - D(1/4) < 0.$$

Town 1's equivalent of H is farther along the same path, giving $m_1^{Q_R}(H) > m_2^{Q_R}(H)$. Hence every anonymous, strongly Pareto evaluation calibrated on Q_R ranks p^1 above p^2. On the capacity-first path, untreated positions tie and $q_2 > q_1$, so every such evaluation ranks p^2 above p^1. Thus neither direction survives all admissible shared paths.

The derivative gap is uniformly at least $1/50$. The stated equivalent-order and witness inequalities are strict and independent of R, and the equivalent roots lie below the smallest budget. These inequalities therefore have positive uniform margins on the compact budget interval. Consequently they persist for sufficiently small C^1 changes in g_1, g_2, with $g_i(0) = 0$, keeping preferences quasilinear in specialist care. The adjusted witness points retain strictly increasing coordinates; the equivalent roots vary continuously because the derivatives remain positive. This proves the stated stability. Separate calibration is used at each R; no common scale across different endowments is asserted. □

O.7.2. A concavity boundary for reference reversal

The reversal in Example O.1 uses smooth quasilinear preferences that are not globally concave in general care. The next result explains why this choice matters. Within a concave quasilinear class, the strict complementary-allocation condition needed for the efficient boundary paths already implies the stronger same-bundle sorting condition. This boundary concerns that class, not all concave multidimensional preferences.

Proposition O.8 (Concavity links complementary and same-bundle sorting)**.** *On $[0, R] \times [0, 1]$, let $u_i(s, z) = g_i(s) + z$, where g_i is C^1, concave, and $g_i' > 0$. If strict OCA holds with town 1 relatively favoring general care, then OSC holds. In particular, efficient-reference calibration cannot produce the strict reference reversal in Example O.1 within this class.*

Proof. First apply OCA to the bundles $x = (R-\varepsilon, d_\varepsilon)$ and $y = (R, 0)$, where $d_\varepsilon = g_1(R) - g_1(R-\varepsilon) \in (0, 1)$ for small $\varepsilon > 0$. Town 1 is indifferent; town 2's complementary allocation must strictly improve. Hence

$$g_1(R) - g_1(R - \varepsilon) > g_2(\varepsilon) - g_2(0).$$

Divide by ε and take the limit to get $g_1'(R) \geq g_2'(0)$. Concavity implies, for every $s \in [0, R]$,

$$g_1'(s) \geq g_1'(R) \geq g_2'(0) \geq g_2'(s).$$

We show that the integral inequality is strict on every interval of positive length. Otherwise, for some $a < b$, $g_1(b) - g_1(a) = g_2(b) - g_2(a)$. Continuity and the derivative ordering imply that both derivatives equal a common constant $c = g_1'(R) = g_2'(0) > 0$ on $[a, b]$. Concavity then forces $g_1' = c$ on $[a, R]$ and $g_2' = c$ on $[0, b]$. Choose $0 < \varepsilon < \min\{R - a, b, 1/c\}$. The bundles $(R - \varepsilon, c\varepsilon)$ and $(R, 0)$ leave town 1 indifferent, and their complementary bundles also leave town 2 indifferent, contradicting strict OCA. Thus

$$g_1(t) - g_1(s) > g_2(t) - g_2(s) \qquad (s < t).$$

If town 1 is indifferent to replacing specialist care by more general care, town 2 strictly prefers the original bundle. If town 2 is indifferent to the same replacement, town 1 strictly prefers the new bundle. These are the two OSC implications. Theorem 2 then rules out a shared path that reverses an EC-robust weak comparison. A path may still tie positions that EC ranks strictly, as explained after that theorem; the conclusion here excludes the opposite strict ranking in the example. □

The proposition is a scope check, not a general defense of concavity or an impossibility for nonseparable preferences. It also does not remove dependence on the reference endowment: the conclusion is conditional on OCA at the declared budget.

O.7.3. Pooling more specialist resources

The main application protects facilities assigned to outsiders. The next result changes only that convention. It maintains the strongest sorting assumptions and asks whether the fixed physical policy menu can still be ranked when the hypothetical pair may share more than one specialist facility.

Proposition O.9 (A pooled specialization endowment can remove every assignment ranking)**.** *Fix $n \geq 2$, common physical bundles $L = (a, 0)$, $H = (a, 1)$, and reference endowment (C, B), with $0 < a < C$ and $B \geq 1$. Suppose all pairs have the full bilateral sharing problem for this endowment and satisfy strong monotonicity, P-OCA, and P-OSC after normalization. For $1 \leq K < n$, compare assignments of H to K recipients and L to the rest. If $B = 1$, the sorted-recipient comparison of Theorem 4 and the top-K conclusion of Section O.4.2 hold. If $B > 1$, no two distinct assignments are universally weakly ranked in either direction by anonymous EC social preferences.*

Proof. Normalize the reference endowment (C, B) to the unit square and set $b = a/C \in (0, 1)$. The actual untreated bundle is $L_b = (b, 0)$, and the treated bundle is $H = (b, 1/B)$. If $B = 1$, these bundles lie on the bottom and top efficient boundaries. Section O.4.2 derives the strict opposite state rankings for every $0 < b < 1$, and the sorted-list matching argument of Theorem 4 applies. It depends on the state bundles lying on those boundaries, not on the narrower baseline-accounting bound in the main application.

Suppose now that $B > 1$. The actual treated bundle lies in the interior of the reference square. Fix different recipients $i < j$, and denote town i's indifferent representatives of H on the two efficient paths by $q_i^-(H)$ and $q_i^+(H)$. Theorem 2 says that these are the unique minimum and maximum of town j's evaluation on town i's contour. Since H is on neither boundary path,

$$q_i^+(H) \succ_j H \succ_j q_i^-(H).$$

Town j's indifferent representative therefore lies farther along E^- than town i's and less far along E^+. With common increasing parametrizations,

$$m_i^-(H) < m_j^-(H), \qquad m_i^+(H) > m_j^+(H).$$

The two coordinates disagree, so different recipients' treated positions are EC-incomparable.

Every untreated position has the same lower-path score λ. Each town strictly prefers H to L_b, so every treated position has a lower-path score strictly above λ. An untreated position consequently cannot dominate any treated position. A favorable perfect matching between two policies with exactly K treated positions must therefore pair all rival treated positions with proposed-winner treated positions. All K proposed-winner treated positions are needed for those matches; none can instead be used for an untreated rival position.

Different recipients' treated positions are incomparable, so the only possible treated match is the same person's treated position. Two distinct K-recipient sets cannot be matched in this way. Theorem 3 therefore rules out universal weak ranking in either direction. □

The change concerns only the hypothetical reference institution. Actual recipients still receive one facility. When outsiders' facilities enter the pool, a treated bundle no longer occupies the efficient top boundary, and the two reference calibrations disagree about its recipients' relative welfare. This shows why the one-facility convention is substantive. It does not show that every change in general-care capacity overturns a ranking, nor does it select the ethically appropriate pool.

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