Do Parties Solve Arrow's Problem?

Aggregation over an endogenous political offer

Niels Boissonnet*

8 November 2026

Abstract

In Arrow's theorem the options are given. In an election they are not: parties form, and citizens then rank them. Can party formation make preferences over parties consistent? I take a continuum of citizens in the plane and a mechanism that produces a finite offer of parties from each distribution. The answer turns on second choices: each duel between two parties is decided by the members of the others, according to which of the two they prefer. When parties sit at the mean of their members, these choices are free, and no anonymous, independent, non-imposed and non-null rule exists. For three parties whose producible vote shares form a convex set, a rule that ranks each pair strictly both ways on a range of shares exists if and only if a rule with one quota per pair does, so consistency alone yields a monotone rule. With four parties it does not: on shares that a fixed offer produces, such a rule exists but no monotone one. A mechanism escapes when it offers two parties, aligns them, or keeps them far apart, with exact ties broken by a suitable arbiter. Letting a party survive only when its existence changes the election screens the cycles under plurality without noise and three births, but mostly leaves one party. With three births, once every party has a chance of winning or a weight in the outcome, the impossibility returns. Parties solve Arrow's problem only when they constrain the second choices of their members.

1 Introduction

Social choice theory studies aggregation rules on a given set of options. The impossibility theorems of Arrow (1951) and his successors take the political offer as given: the rule applies to whatever options are on the table, and the options do not depend on the preferences. Political science has long known otherwise. The offer is produced by party competition, so it is a function of the citizens' preferences, and this restricts the domain on which an aggregation rule must work. Its shape also depends on the electoral rule: plurality rule tends to produce two parties, and proportional representation many (Duverger, 1954; Cox, 1997). For this reason, Riker (1982) suggested that two-party systems rescue majority rule, since between two options there is always a majority winner. Aldrich (1995) proposed that parties exist to tame majority cycles, and Sen (1999, p. 354) suggested that when people cluster in parties, inconsistencies "can yield ground".

In this paper, I study whether the impossibility theorem still applies when the political offer is generated by a very general process of party formation in a spatial voting model. Citizens form a continuum in the plane. A competition mechanism maps each distribution of citizens to a finite offer of parties, the citizens rank the parties by distance, and a rule must

*Proofs are in the appendices; a Supplementary Appendix proves the results of Section 7, the variants of Section 8, the second half of Proposition 6, and complements to Sections 5 and 6. A companion paper (Boissonnet, 2026) studies the size of majority cycles among parties in given societies; the hexagon of Section 4.3 and the rectangle of Section 6 are taken from it.

rank the parties on every offer that the mechanism produces. The mechanism is arbitrary, with one requirement: the offer depends on the distribution of the citizens, not on their names.

The impossibility still applies as long as the citizens' second choices are free. A citizen's second choice is the party ranked just after the nearest one. Take a left, a right and a centre party, each at the centre of gravity of its members, the citizens closest to it, with the centre party a little above the axis that joins the two others. In the duel between left and right, the members of the left party vote left and those of the right party vote right. So the members of the centre party decide, according to which of the two is their second choice. In the same way, the members of the left party decide the duel between centre and right, and those of the right party the duel between left and centre. Each duel is arbitrated by the members of the third party.

When parties sit at the mean of their members, nothing fixes how these members split. One can move members of the centre party across the line that separates the citizens closer to the left from those closer to the right, while keeping their centre of gravity, hence the offer, unchanged. If the society spreads widely enough around the point equidistant from the three parties, the members of the left and right parties can be moved in the same way. The three duels then move independently, and all the vote shares around a general tie can be produced. Cycles are among them. With three parties of equal size, if most centre members prefer the left to the right, most left members the right to the centre, and most right members the centre to the left, then left beats right, right beats centre and centre beats left. Since the shares move independently, it is not only majority rule that fails: no rule survives. Parties solve Arrow's problem only when they constrain the second choices of their members.

Firstly, I prove that impossibility extends to this case (Section 4). An anonymous, independent and non-imposed rule must declare every pair of parties that belongs to a "star" triple indifferent: a triple on which the competition can produce, around a common point (in the applications, the point where every duel is tied), all the vote shares it produces anywhere. When parties sit at the mean of their members, every pair belongs to a star triple, so no non-null rule exists; this holds for any number $k \geq 3$ of parties, whether all equilibria are admitted or only the best one, and also for parties subject to a minimum size. With three parties, the triple is a star triple for every mechanism that sees each electorate only through its size and its mean, as long as the sizes it produces form a convex set that treats the parties alike and the share of a party in a duel stays strictly above the smallest size a party can have (Proposition 1). Pareto plays no role, since in both examples no pair of parties is ever unanimous; Wilson's non-imposition takes its place. The proof is short once anonymity and independence have reduced the problem to vote shares (Lemma 1). The constructions use societies of small, distant groups, but the failure of majority rule does not depend on them: it already cycles in a society uniform on a convex hexagon (Section 4.3).

Secondly, with three parties, a rule exists only if a very simple one does: the *quota rule* (Section 5). It fixes a threshold for each pair of parties: when the share of citizens preferring one party to the other is above the threshold, the pair is ranked one way; when it is below, the other way, with a convention at the threshold itself. Majority rule, with a convention for exact ties, is the quota rule whose thresholds are all one half. A quota rule works when the competition can never produce shares that put the three pairs in a cycle, such as left above right, right above centre and centre above left. Precisely, suppose that the vote shares the competition can produce on a triple form a convex set, and that its pairs produce no other share at another offer. Ask the rule to rank each pair strictly one way on some range of shares and strictly the other way on another range, which is a little stronger than Wilson's non-imposition. Then such a rule exists if and only if a quota rule works. The equivalence fails without convexity, or under Wilson's non-imposition. A quota rule is monotone: on each pair, the ranking responds to the share in one direction only. So, with three parties, consistency alone guarantees that a monotone rule exists whenever some rule does. This does

not extend to four parties. A fixed offer of four platforms in the plane produces a convex set of shares on which a rule of this kind exists, but no monotone one, and hence no quota rule. The rule that works ranks one pair one way for middle shares and the other way for low and high shares. With four parties, monotonicity has to be assumed. The characterization does survive, for any number of parties, on the classical restricted domains, where the shares are those of a fixed family of strict rankings. It then holds on the relative interior of the domain, when the shares of every triple span three dimensions.

Thirdly, parties do solve the problem when they stand far apart relative to the electorate (Section 6). Take three parties that are not aligned, and the point equidistant from them. Suppose that this point never lies inside the convex hull of the electorate, and that every party has members. Then one party of the triple, say the centre, is never ranked last. The members of the left and right parties all rank it second, and only its own members are free in their second choices. The centre can then lose a duel only to a party with more than half of the citizens as members, and that party wins its other duel too, so majority rule has no strict cycle. Exact ties need care. For fixed parties and all the societies that stay clear of that point, no rule exists, because the party that is never last depends on the side of the point where the society lies. The remedy is to break exact ties in favour of the party preferred by a fixed arbiter, placed on the society's side of the point. Majority rule is then consistent. Two parties always escape, and so do aligned parties, since preferences over them are single-peaked. With parties at the mean of their members, many parties need a long and thin society: on a uniform rectangle, the optimal offer of k parties is aligned if and only if the length exceeds a threshold between $0.56\,k$ and $0.70\,k$ times the width (Boissonnet, 2026). The companion paper also measures how large the cycles are when they survive.

Fourthly, I let the electoral rule shape the offer, as Duverger's law says it does (Section 7). Proto-parties are born at fixed positions, and one survives only if enough citizens gain from its existence. With three proto-parties under plurality without noise, this screens the cycles. A losing party survives only as a spoiler: without it, the other loser would win. So the winner of a surviving triple loses both duels, and majority rule has no cycle. The price is high. The two-party system that Duverger's law predicts never forms, mostly a single party survives, and the screening fails with four proto-parties. The escape closes as soon as every party weighs on the expected outcome, by chance under noise, or by proportion under a compromise between platforms weighted by votes. With three proto-parties, the triple is then a star triple again and no rule exists. Distant citizens are not needed when support costs nothing: under plurality and runoff with noise, this holds on every disc centred at the point equidistant from the three parties, and under the compromise on every disc whose radius exceeds a bound.

Related literature: Against Riker, Mackie (2003) argues that cycles are rare; Barbaro and Kurella (2026) find no robust majority cycle in 253 national elections, since the only cycle in their samples, in Peru's 2011 presidential election, does not survive their robustness checks; Lachat and Laslier (2024) also find cycles to be very rare, and Desai and Kalandrakis (2025) find a party in the core of almost every party system. That majority cycles are everywhere in a multidimensional spatial model is old (Plott, 1967; McKelvey, 1976); Tullock (1967) argued that, with many voters, they are confined to a small central region. And deliberative democrats propose another route to coherence: deliberation may bring preferences close to single-peakedness (Dryzek and List, 2003), a prediction that List et al. (2013) find supported in deliberative polls. Arrow's theorem has been studied on restricted domains, including Euclidean preferences in the plane (Border, 1984), but always with a fixed set of options. Social choice on variable agendas (Grether and Plott, 1982; Denicolò, 1985) lets the rule depend on the agenda, but the agenda is an exogenous argument of the rule. The theory of strategic candidacy (Dutta et al., 2001) lets candidates enter or withdraw, but the set

of potential candidates is given. Iwata (2016) lets a nomination stage select the eligible alternatives; there the opinions of the nominators, not the positions of the citizens, decide which alternatives enter. In citizen-candidate models (Osborne and Slivinski, 1996; Besley and Coate, 1997), a candidate may enter only to spoil the victory of another, the mechanism that drives Section 7; Feddersen et al. (1990) study entry under plurality rule with strategic voters. These models study equilibrium platforms, not aggregation over them. To my knowledge, no Arrovian result treats an agenda that is a function of the voters' profile, with independence linking profiles across agendas.

Arrovian theorems on economic and spatial domains (Kalai and Muller, 1977; Kalai et al., 1979; Border, 1983, 1984; Ehlers and Storcken, 2008; Le Breton and Weymark, 2011) prove impossibility or characterize possibility for a fixed set of alternatives; Wilson (1972) removes Pareto, and Border (1983) proves a Wilson-type trichotomy for private-goods economies. My domain differs in that the alternatives are a function of the profile, and independence links only profiles whose offers share a pair. On fixed families of rankings, my quota result meets the work on Condorcet domains (Inada, 1969; Moulin, 1984; Sethuraman et al., 2006; Dasgupta and Maskin, 2008; Nehring and Puppe, 2007, 2010; Puppe and Slinko, 2026), which assumes neutrality or monotonicity; Section 5 makes the comparison. Models of party formation (Gomberg et al., 2004; Caplin and Nalebuff, 1997; Laver and Sergenti, 2012; Levy, 2004; Morelli, 2004) produce an offer from a distribution of citizens; parties at the mean of their members appear in Gomberg et al. (2004) and Laver and Sergenti (2012). None of these papers studies aggregation over the offer. Finally, Gomberg et al. (2005) prove that, with a continuum of voters and a rule that depends only on the distribution of types, no Arrovian rule exists; their swap argument is the one I use to reduce the problem to vote shares, and I extend it to a profile-dependent offer and to non-imposition in place of Pareto.

2 Model

There is a continuum of citizens, indexed by $i \in I = [0,1]$ with Lebesgue measure λ. A *profile* is a measurable map $x : I \to \mathbb{R}^2$; x_i is citizen i's ideal point. Write μ_x for the distribution of x. I assume throughout that μ_x puts no mass on any line; Section 8 studies what changes without this assumption. Any such distribution may be considered: this is the counterpart of Arrow's unrestricted domain.

A *competition mechanism* assigns to each profile a set of *offers*, each a finite set $O \subset \mathbb{R}^2$ of platforms with $|O| \geq 2$. A profile may receive several offers (several equilibria) or none (a mechanism may require moments, or produce no stable offer); the domain $\mathcal{D}$ is the set of pairs (x, O) such that O is an offer at x, and nothing below requires every profile to receive one. The only requirement is that the mechanism ignores the names of the citizens.

Assumption 1 (Offers depend on the distribution alone)**.** If $(x, O) \in \mathcal{D}$ and $\mu_y = \mu_x$, then $(y, O) \in \mathcal{D}$.

Two mechanisms serve as running examples.

Example 1 (Parties at the mean of their members)**.** Fix $k \geq 3$. Each citizen joins the party whose platform is closest to her, and each party stands at the mean position of its members. An *equilibrium offer* is a set of k platforms that satisfies both conditions, every party having members of positive mass. One may admit all equilibrium offers, or only the *optimal* one, which minimizes the mean squared distance between citizens and their closest platform. For the means to exist I assume that μ_x has a finite first moment, and a finite second moment for the optimal offer.

Example 2 (Minimum size)**.** Fix $k \geq 3$ and $\varepsilon \in (0, 1/k]$. Any set of k platforms is an offer, provided each platform attracts at least a share ε of the citizens. This is the reduced form of an electoral threshold.

Citizen i ranks the offered platforms by Euclidean distance to x_i. Since μ_x charges no line, almost every citizen has a strict ranking. For $a \neq b$ in O, let $s_{ab}(x) = \lambda\{i : |x_i - a| < |x_i - b|\}$ be the share of citizens who prefer a to b; then $s_{ba} = 1 - s_{ab}$. An *aggregation rule* F assigns to each $(x, O) \in \mathcal{D}$ a weak order $F(x, O)$ on O. In words, the rule ranks the parties that are offered, and it does so for every society and every offer the competition can produce.

The rules I am interested in satisfy the following axioms.

Axiom 1 (Anonymity)**.** If $\sigma : I \to I$ is a measure-preserving bijection[1] and $(x, O) \in \mathcal{D}$, then $F(x \circ \sigma, O) = F(x, O)$.

Axiom 2 (Independence)**.** Let $(x, O), (y, O') \in \mathcal{D}$ and $a, b \in O \cap O'$. If the sets $\{i : a \succ_i b\}$ under x and under y coincide up to a null set, and likewise the sets $\{i : b \succ_i a\}$, then $F(x, O)$ and $F(y, O')$ rank a and b in the same way.

This is essentially Arrow's independence of irrelevant alternatives, read on a variable offer: the ranking of two parties depends only on which citizens prefer which of the two. Arrow's reason is an election in which a candidate dies after the ballots are cast. The winner should be found by deleting that name from the ballots; otherwise the result depends on the accident of whether the death came before or after the vote (Arrow, 1950, p. 337). The axiom compares profiles whose offers may differ: otherwise a mechanism whose offer changes with every profile would make it empty.

Axiom 3 (Non-imposition)**.** For every ordered pair (a, b) offered together at some profile, there is $(x, O) \in \mathcal{D}$ with $a, b \in O$ such that a is ranked weakly above b by $F(x, O)$.

Axiom 4 (Non-nullness)**.** Some $F(x, O)$ ranks some pair strictly.

Non-imposition is Wilson's (1972). I use it instead of Pareto for a reason specific to this setting. When the platform of a party is the mean of its members, as in Example 1, a positive mass of members strictly prefers their party to any other platform.[2] No offered pair is ever unanimous, Pareto never applies, and a rule that ranks the parties by their first coordinate, whatever the preferences, satisfies anonymity, independence, Pareto and non-dictatorship.[3] Non-imposition is the weakest standard axiom that rules out such imposed rules: it only asks that each pair can be ranked both ways. Wilson's theorem says that on the unrestricted domain, with a finite electorate, independence and non-imposition leave only the null rule, a dictator or an inverse dictator. With a continuum and anonymity there is no dictator to fall back on, and the conclusion of an impossibility theorem becomes that no rule exists at all.

[1] A measure-preserving bijection is a relabelling of the citizens that keeps the size of every group: each label goes to exactly one citizen, and every measurable set of citizens has the same measure before and after the relabelling. For instance, $\sigma(i) = i + \frac{1}{2}$ for $i < \frac{1}{2}$ and $\sigma(i) = i - \frac{1}{2}$ for $i \geq \frac{1}{2}$ exchanges the two halves of I. The profile $x \circ \sigma$ gives citizen i the ideal point of citizen $\sigma(i)$: the same positions are occupied by the same mass of citizens, under other names.

[2] Let p be the mean of the members and $q \neq p$. The mean of $m - p$ over the members is zero, so a positive mass of members satisfies $(m - p) \cdot (q - p) \leq 0$, hence $|m - q|^2 - |m - p|^2 = |q - p|^2 - 2(m - p) \cdot (q - p) > 0$.

[3] The rule ignores preferences, so it is anonymous and independent. It is not dictatorial: if $a_1 > b_1$ in an offer, moving one citizen to b changes neither the distribution nor the offer, and she prefers b while the rule ranks a above.

3 Reduction to vote shares

The first step borrows an idea from Gomberg et al. (2005): with a continuum of anonymous citizens, independence can only see how many citizens prefer a to b.[4]

Lemma 1 (Reduction)**.** *Let F satisfy anonymity and independence. Let $(x, O), (y, O') \in \mathcal{D}$ with $a, b \in O \cap O'$. If $s_{ab}(x) = s_{ab}(y)$, then $F(x, O)$ and $F(y, O')$ rank a and b in the same way.*

Proof. Write P_x and P_y for the sets of citizens who prefer a to b under x and under y. They have the same measure s_{ab}, and since almost no citizen is indifferent, their complements are, up to null sets, the sets of citizens who prefer b to a. By the isomorphism theorem for measures,[5] there is a measure-preserving bijection σ of I that sends P_y into P_x and its complement into the complement of P_x, up to a null set. Let $z = x \circ \sigma$: citizen i receives the ideal point of citizen $\sigma(i)$. Then $\mu_z = \mu_x$, so $(z, O) \in \mathcal{D}$ by Assumption 1, and anonymity gives $F(z, O) = F(x, O)$. Citizen i prefers a to b under z if and only if $\sigma(i) \in P_x$, that is, for almost every i, if and only if $i \in P_y$. So, up to a null set, the same citizens prefer a to b under z and under y, and the same citizens prefer b to a. Independence gives the same ranking of a and b under (z, O) and (y, O'), hence under (x, O) and (y, O'). □

The lemma uses Assumption 1: the relabelled profile z must still be admitted, with the same offer. It also uses independence across offers, since O' may differ from O. It uses neither the transitivity of $F(x, O)$ nor the geometry, and it holds when lines may carry mass, with the pair (s_{ab}, s_{ba}) in place of s_{ab} (Lemma 4).

By the lemma, all the information a rule needs about a pair (a, b) is given by the share s_{ab} of citizens who prefer a to b. Call a share $w \in [0, 1]$ *producible* for the pair if $w = s_{ab}(x)$ for some $(x, O) \in \mathcal{D}$ with $a, b \in O$. A rule is then described by a *labelling*: for each pair (a, b) and each producible share w, a label $\ell_{ab}(w)$ equal to F, S or I, according as the rule ranks a above b, ranks b above a, or ties them when the share is w. Majority rule, for instance, has $\ell_{ab}(w) = \mathsf{F}$ for $w > 1/2$ and $\ell_{ab}(w) = \mathsf{S}$ for $w < 1/2$. The label may depend on the pair, that is on the positions of a and b, but on nothing else: not on the other parties offered, nor on how the citizens are spread. Two consequences follow. The labels of (a, b) and (b, a) carry the same information: since $s_{ba} = 1 - s_{ab}$, $\ell_{ba}(1 - w)$ is $\ell_{ab}(w)$ with F and S exchanged. And since the distribution fixes every share, $F(x, O)$ depends only on (μ_x, O).

A complete ranking of the parties is transitive if and only if it is transitive on every triple, so it is enough to look at triples. This occurs when the labelling is *valid* on the triple: at every $v \in V$, the three labels $\ell_{ab}(v_1)$, $\ell_{bc}(v_2)$, $\ell_{ca}(v_3)$ rank a, b, c transitively: either all are I, or at least one is F and at least one is S (Lemma 5). For three parties a, b, c, write $v = (s_{ab}, s_{bc}, s_{ca})$, and let V be the set of *producible* vectors $v(x)$, over all $(x, O) \in \mathcal{D}$ with $a, b, c \in O$. Read the labels in the same cyclic order, so that F is a step forward along $a \to b \to c \to a$ and S a step backward. Otherwise, going around the cycle puts a party strictly above itself.

[4] Their anonymity axiom requires the rule to depend only on the distribution of preferences. In the present paper, it has to be invariant under relabellings of the citizens, and the dependence on the distribution is a conclusion. Their argument uses profiles with two preference types. Here the offer depends on the distribution, so the relabelled profile must keep the whole distribution of the original one, which requires the isomorphism theorem for measures (footnote 5). Independence also links different offers, and the conclusion bears on the shares of a pair, without Pareto, rather than on decisive coalitions.

[5] Apply the isomorphism theorem for measures (Halmos and von Neumann, 1942; Kechris, 1995, Theorem 17.41) to Borel subsets of full measure of the pieces, chosen to avoid the Cantor set K: it matches the pieces outside a null Borel set. The two leftover null sets contain K, so they are uncountable, and the Borel isomorphism theorem (Kechris, 1995, Theorem 15.6) matches them. The glued map is a bijection of I, Borel both ways, that preserves λ on Borel sets, hence on Σ by completeness. The same theorem shows that the choice $I = [0, 1]$ entails no loss of generality.

Two short lemmas carry the whole impossibility. The first is elementary: on a product of share sets, a valid rule has a rigid form.

Lemma 2 (Products)**.** *Let $X \times Y \times Z$ be a set of share vectors on which a rule is valid, and suppose the label of the first coordinate takes two different values on X. Then the labels of the second and third coordinates are constant on Y and Z, one equal to* F *and the other to* S*.*

Proof. Every combination of a label taken on X, one on Y and one on Z occurs on the product, so it comes from a weak order. Let $\lambda \neq \lambda'$ be two labels taken on X; one is strict, say $\lambda = \mathsf{F}$ (the case S is symmetric), and λ' is S or I. Let (m_2, m_3) be labels taken on Y and Z. With $\lambda = \mathsf{F}$, the pair (m_2, m_3) must contain an S. With $\lambda' = \mathsf{S}$ it must also contain an F; with $\lambda' = \mathsf{I}$ it must be (I, I) or strict and opposite. Either way (m_2, m_3) is (F, S) or (S, F). Both are constant. If Y carried two labels, the product would pair each of them with the same label m_3 of Z, and (m_2, m_3), (m_2', m_3) with $m_2 \neq m_2'$ cannot both be strict and opposite; likewise for Z. □

The second lemma is the engine. Call $h = (1/2, 1/2, 1/2)$ the *centre*: the configuration where every duel is tied.

Lemma 3 (Star)**.** *Let V be the set of producible share vectors on a triple, and D_1, D_2, D_3 its coordinate projections. Suppose there is a point $h \in V$ and a family of products $X_k \times Y_k \times Z_k \subset V$, all containing h, whose factors cover D_1, D_2 and D_3. Then every rule that is valid on V and satisfies non-imposition is null on the triple: all its labels are* I*.*

Proof. Let (e_1, e_2, e_3) be the labels at h. If all three are I, every product of the family contains an I in each factor, so by Lemma 2 no factor carries two labels, and all labels on the covering factors are I. Otherwise some e_i is strict. By non-imposition there is $w \in D_i$ with $\ell_i(w) \neq e_i$, and by the covering some product has w and h_i in its i-th factor. That product is not null, since it contains h, so by Lemma 2 its two other coordinates are constant, strict and opposite; at h, this says that the two labels other than e_i are strict and opposite. This holds for every strict coordinate i. It is impossible: if some label at h is I, pick a strict i, and the two others contain that I; if all three are strict, two are equal, and the third sees two equal labels. □

In words: a rule cannot be strict at the centre and non-imposed elsewhere, because every producible share "sees" the centre through a product. The products come from the way the duels are decided.

Who decides a duel. On a triple $\{a, b, c\}$, call the citizens who rank a first the *members* of a, with mass m_a; likewise for b and c. In the duel between a and b, the members of a and of b vote for their own party, so the members of c decide:

$$s_{ab} = m_a + \alpha_c, \qquad s_{bc} = m_b + \alpha_a, \qquad s_{ca} = m_c + \alpha_b,$$

where $\alpha_c \in [0, m_c]$ is the mass of members of c who prefer a to b, and similarly for the others. At sizes $m = (m_a, m_b, m_c)$, the share vectors lie in the box

$$B(m) = [m_a, m_a + m_c] \times [m_b, m_b + m_a] \times [m_c, m_c + m_b],$$

which contains the centre h if and only if no party has a majority, $\max m \leq 1/2$. *Arbitration is free* at m if the whole box is produced.

Every application below follows one recipe: find sizes without a majority party at which arbitration is free, enough of them for their boxes to cover all the shares of the triple. These boxes are the products of the star lemma, and h is their common point (Corollary 1). The recipe fails in two cases, which are the two escapes. Either some party always holds a majority: if the sizes are always a permutation of $(0.8, 0.1, 0.1)$, that party is a Condorcet winner and majority rule is valid (Proposition 5). Or arbitration is not free: this is Section 6.

4 Impossibility

4.1 The theorem

With k parties, a pair $\{a, b\}$ belongs to $k-2$ triples and may be offered in many configurations. Non-imposition for the pair may then be satisfied in an offer where the pair is accompanied by other platforms. The right hypothesis makes the star lemma apply to *all* shares a pair ever produces.

Definition 1. Let D_{ab} be the set of shares $s_{ab}(x)$ over all $(x, O) \in \mathcal{D}$ with $a, b \in O$, and let V_T be the set of share vectors over all (x, O) with $T = \{a, b, c\} \subset O$. The triple T is a *star triple* if V_T contains products through a common point whose factors cover D_{ab}, D_{bc} and D_{ca}.

Theorem 1 (Impossibility)**.** *Let F satisfy anonymity, independence and non-imposition. Then F is null on every pair that belongs to a star triple.*

Proof. By Lemma 1 the rule is a labelling. Let $\{a, b\}$ be a pair of a star triple T. The factors of the products lie in $\pi_i(V_T) \subset D_{ab}$ and cover D_{ab}, so non-imposition for the pair bears on the covering factors, and Lemma 3 gives $\ell_{ab} \equiv \mathsf{I}$ on D_{ab}. □

A non-null rule can therefore survive only through pairs that are never offered in a star triple, for instance pairs offered only in collinear offers (Supplementary Appendix S1.2). The theorem needs no geometry. The geometry enters when one checks which triples are star triples for a mechanism.

4.2 Mechanisms that see sizes and means

The impossibility rests on three properties of the mechanism: it can offer three non-aligned parties; no party is forced to hold a majority; and the members of a party are free in their second choices. The third holds for parties at the mean of their members, and for every mechanism that sees each electorate only through its size and its mean position. Formally, let the societies of the domain have a finite first moment, and call the *cell* of a platform the set of points strictly closer to it than to any other platform of the offer. The mechanism *reacts only to sizes and means* if

(S) whenever $(x, O) \in \mathcal{D}$ and every cell of O has positive mass, $(y, O) \in \mathcal{D}$ for every profile y whose distribution has a finite first moment and gives each cell of O the same mass and the same mean as μ_x.

For a triple T, let M_T be the set of size vectors of the societies at which T is offered alone, with three cells of positive mass; let $\mu_T = \inf\{m_a : m \in M_T\}$ be the smallest size a party can have there; and write $B^\circ(m)$ for the interior of the box $B(m)$.

Proposition 1 (Offers that react to sizes and means)**.** *Let the societies have a finite first moment, let the mechanism satisfy (S), and let T be a non-aligned triple offered alone at some society.*

(i) $V_T \supseteq B^\circ(m)$ for every $m \in M_T$: the members of each party arbitrate freely, up to the boundary of the box.

(ii) If M_T is convex and invariant under the permutations of the parties, and every share of a pair of T produced at any offer lies strictly between μ_T and $1 - \mu_T$, then T is a star triple.

Theorem 1 then makes every valid rule null on the three pairs of T. The proof moves the members of each party between its two sectors while keeping their mean, and then applies a midpoint argument (Appendix B). Parties at the mean of their members, with three parties and all equilibria admitted, are an instance: an equilibrium offer stays one when each cell keeps its mass and its mean, every size vector with positive entries is produced, so $\mu_T = 0$, and no share is 0 or 1, since each party has members who prefer it to any other platform. Survival rules that read sizes alone get the closed boxes, and are treated directly by Proposition 2. The hypotheses cannot be dropped. Offer T when the sizes are a permutation of $(0.8, 0.1, 0.1)$ and the mean of each party's members lies on the bisector of the two other parties: (S) holds, the members of each party are split between its two sectors, so every share lies strictly between 0.1 and 0.9, but the sizes are not convex, and majority rule is valid. The arc mechanism of Section 6, on the societies that give every party members, produces every size vector of the open simplex, violates (S), and has a valid rule. Mechanisms that react to more than sizes and means, such as survival that depends on transfers of votes (Section 7), are outside the proposition. So are mechanisms that violate the other two properties. The next two sections ask which of them escape.

The proposition looks at a triple offered alone. The two examples of Section 2 meet the hypothesis of Theorem 1 on every pair with any number $k \geq 3$ of parties: parties at the mean of their members, whether all equilibria are admitted or only the optimal offer, and parties subject to a minimum size $\varepsilon \leq 1/k$ (Propositions 6 and 7 in Appendix B). The other $k - 3$ parties are placed far away, and since $k\varepsilon \leq 1$ the sizes of the triple can avoid a majority party. So no non-null rule exists in either example. The mean-of-members condition pins down one point of each electorate and leaves the second choices free; this is the precise sense in which endogeneity does not help. A reduced form of an electoral threshold does not help either.

4.3 Interpretation of the theorem

Why a weak order? The theorem asks the rule for a ranking of the offered parties, as Arrow does. Weaker demands make the problem disappear, for a reason that has nothing to do with parties. Let a party beat another when more than a share $(k-1)/k$ of the citizens prefers it, where k is the largest number of parties ever offered. This strict relation never cycles, on any offer and any domain. Along a cycle of length $L \leq k$, each citizen agrees with at most $L - 1$ of the L duels, so the shares along the cycle sum to at most $L - 1$ and cannot all exceed $(L-1)/L$. This is the threshold of Greenberg (1979) for k alternatives; with a continuum of alternatives and concave densities, Caplin and Nalebuff (1988) bring it down to 64 per cent. The relation is acyclic, but its indifference is not transitive: two parties may each tie with a third and still be ranked strictly.

Asking only for a quasi-transitive relation, whose strict part is transitive, does not help either under Wilson's non-imposition. Fix one pair of platforms, rank it by majority whenever it is offered, and declare every other pair indifferent. This rule is anonymous, independent, non-imposed, non-null and quasi-transitive. It is also useless: it ranks one pair and is silent on all others. The demand for a weak order is what makes the question sharp: can the citizens' preferences over the parties be summarized by a ranking? Whether quasi-transitivity with the strong robust non-imposition of Section 5 still yields an impossibility is open.

Unimodal societies. The constructions behind Propositions 6 and 7 use societies made of small, distant groups. One may suspect that the problem lives only there. For majority rule, it does not. Let the citizens be uniform on a convex hexagon that is unchanged by a rotation of a third of a turn, but by no reflection. The optimal offer of three parties at the mean of their members can then be a "pinwheel": the bisector of two parties cuts the electorate of the third along its centre of gravity, but not into two equal halves, so each electorate leans towards

the next party. Every duel is then won with the same share above one half, and majority cycles (Boissonnet, 2026, Proposition 1). So the failure of majority rule does not need distant groups. Whether some rule other than majority is valid on such societies is open.

The theorem is a confirmation of Arrow in a setting where Arrow's hypotheses fail: the domain is small, Pareto is vacuous, and the options move with the preferences. It is not a corollary. An impossibility on a large domain says nothing about a smaller one, as the single-peaked domain shows, and the proof uses the specific way in which the mean-of-members condition leaves second choices free.

5 Three parties: an exact characterization

When does a rule exist? For three parties the answer is complete when the producible share vectors of the triple form a convex set and its pairs are offered in no other configuration, under a slightly stronger non-imposition that rules out degenerate rules.

Axiom 5 (Strong robust non-imposition)**.** For every ordered pair (a, b), the set of shares $w \in D_{ab}$ at which $\ell_{ab}(w) = \mathsf{F}$ contains a non-degenerate interval, and so does the set at which $\ell_{ab}(w) = \mathsf{S}$.

In words, each pair can be ranked strictly either way, on a range of configurations and not at an isolated one. Wilson's non-imposition is satisfied by a single profile, and this admits rules that are indifferent except at the ends of the share ranges (Supplementary Appendix S4), or strict everywhere and imposed except at one extreme share (Example S6). Impossibility results are stated under Wilson's axiom, the weakest; the characterization is stated under the strong one.

Definition 2 (Quota rule)**.** A *quota rule* on a triple picks, for each pair, a threshold t_{ab} in the interior of D_{ab}, an orientation $\sigma_{ab} \in \{+1, -1\}$ and a tie convention ($\geq$ or $>$), and ranks a above b when $\sigma_{ab}(s_{ab} - t_{ab})$ is positive (or non-negative), and b above a otherwise.

Majority rule, with each exact tie broken by a fixed convention for the pair, is the quota rule with all thresholds at $1/2$ and all orientations positive. A quota rule is valid if and only if no producible share vector lies in one of the two *cyclic orthants*, above all three thresholds in the forward direction or below all three.

Theorem 2 (Quotas)**.** *Let V be the set of producible share vectors of a triple, convex and non-empty, and suppose that no pair of the triple produces another share at another offer, so that $D_i = \pi_i(V)$. A rule valid on V with strong robust non-imposition exists if and only if a valid quota rule exists.*

Idea of the proof. A quota rule satisfies the axioms. For the converse, suppose V has non-empty interior U (the degenerate case is easier, Appendix C). By Lemma 2, on any small box in U at most one coordinate changes label, and when it does the two others are strict and opposite. Let K_i be the set of *change points* of coordinate i, where its label is not locally constant; strong non-imposition makes each K_i non-empty. Above a change point of coordinate i, the labels of the two other coordinates are constant along the whole fibre of U, so the three axis-parallel lines through any $t \in K_1 \times K_2 \times K_3$ miss U. A convex open set that misses these three lines meets exactly four of the eight orthants at t, forming either a claw or a path in the graph of the cube. A claw can always be turned into a path by moving one threshold to another change point. A path leaves two opposite orthants empty, and these are the cyclic orthants of a quota rule with thresholds t. The quota is valid on the closure by a barycentre argument, with the tie convention chosen on one coordinate (Appendix C). □

The theorem does not say that a valid rule *is* a quota rule: labels may change several times on a range where the two other duels are already decided in opposite directions. It says that whenever a rule exists, a quota rule exists too. Three remarks on its scope.

First, the tie conventions matter. With three arbitrating groups unanimous in the cyclic direction, the shares satisfy $s_{ab} + s_{bc} + s_{ca} = 2$; the quota $2/3$ on every pair with the same tie convention cycles at $(2/3, 2/3, 2/3)$, while $\geq$ on two pairs and $>$ on the third does not. Second, when the producible shares are convex, the characterization reads as a condition on the competition: a rule exists if and only if the configurations of vote shares that the competition can produce leave two opposite cyclic orthants empty. Free arbitration fills a neighbourhood of the centre and leaves none. Convexity cannot be dropped: three boxes inside $[0.4, 0.6]^3$, hence producible by citizens, carry a rule but no quota rule (Remark 2). Third, the statement fails beyond three parties. With four parties there is a convex set of share vectors, produced in the plane by a fixed offer of four platforms, on which a valid rule with strong robust non-imposition exists although no quota rule, with any thresholds, orientations and conventions, is valid even on the relative interior (Proposition 10; the absence of a quota is certified by exact rational arithmetic). The rule has a "band" on one pair, ranking it one way for middle shares and the other way for low and high shares, and the band is forced by two different triples. Producibility is a weak restriction here. Moving every share towards one half by the same factor changes neither the rule, nor the axioms, nor the quota rules, and it brings any convex set close enough to the centre for every point to be the shares of some distribution of citizens over rankings (Lemma 18); the remark on convexity uses the same device. That a fixed offer in the plane produces the set is then checked directly.

The band shows what fails. Call a rule *monotone* if on each pair the ranking responds to the share in one direction: for some orientation, $a \gtrsim b$ at a share implies $a \succ b$ at every share beyond it, as in May's positive responsiveness (May, 1952). A monotone rule with strong robust non-imposition ranks each pair by a single threshold, and differs from a quota rule at most by an indifference at the threshold itself. Quota rules are monotone; the band rule is not, and on its set no monotone rule with strong robust non-imposition is valid. So Theorem 2 says something that does not survive beyond three parties: there, consistency alone guarantees that a monotone rule exists whenever some rule does. With four parties, monotonicity has to be assumed.

On the classical restricted domains, consistency still does the work. Suppose the producible shares are those of the societies whose rankings lie in a fixed family of strict rankings, and that the shares of every triple span three dimensions. Then a valid rule with strong robust non-imposition forces some party of each triple to be never first, or never last, in the family (these are two of the three forms of the value restriction of Sen, 1966; never last gives the windows of Section 6), and the quota rule whose thresholds are the rule's own change points, with the natural orientation, is valid on the relative interior, for any number of parties (Proposition 11). With four parties, a family carries a rule if and only if it carries a quota rule, on the closed set as on the relative interior, triples of lower dimension included (Remark 3(iii)).

Condorcet domains. This result has close neighbours. A family of rankings on which majority never cycles is a *Condorcet domain*: no triple carries a "Latin square", three rankings ijl, jli, lij (Ward, 1965; Sen, 1966; Inada, 1969). In the terms of Puppe and Slinko (2026), the families of the proposition are the copious peak-pit domains, those with four rankings on each triple and a party never first or never last. Dasgupta and Maskin (2008) also take a continuum of voters. They show that if an anonymous and neutral rule satisfying Pareto is consistent on a family, except at finitely many values of the shares, then the family has no Latin square, so majority is consistent on it too. Proposition 11 and Remark 3(i) give the same conclusion with neither neutrality nor Pareto, which are replaced by strong robust

non-imposition. Nor do they need an exceptional value: on the relative interior, majority with exact ties left as indifference is a weak order. The troublesome ties arise only on the boundary, where a ranking has no mass (Remark 3). With ties left as indifference on the whole domain, transitivity needs the stronger extremal restriction of Sen and Pattanaik (1969). But the ties have a simple remedy. On any Condorcet domain without a unanimous pair, majority with exact ties broken by a fixed ranking of the family is valid on the closed set, for any number of parties and whatever the dimensions (Proposition 12). The reason is short: two of the three relations of any cycle are incompatible on the family, so their shares add up to at most one; both reach one half only as exact ties, and no ranking of the family can decide both ties in favour of the cycle. With finitely many voters this remedy is well known: for an even number of voters, a fixed ranking of the domain acts as a dummy voter (Sethuraman et al., 2006, footnote 3). So a quota rule always exists on these domains, and thresholds other than one half are needed only when a triple carries a Latin square.

Dropping neutrality has a price: the hypothesis on dimensions. A Latin square alone is flat, and the quota 2/3 of the first remark above is valid on it, while majority cycles. That rule is anonymous and independent, and it satisfies Pareto; it only treats the two sides of a pair differently. So a Latin square alone excludes neutral rules, not all rules. A fourth ranking, which gives it a third dimension, excludes them all. Non-imposition draws the other line. With independence in place of neutrality, a consistency requirement on ties, Pareto and exact transitivity, the most robust rule is unanimity with an order of precedence: a goes above b when a comes first in a fixed order, unless everyone prefers b (Dasgupta and Maskin, 2003). It works on triples with five rankings, where the proposition finds no rule. There is no contradiction. Inside the domain this rule listens to no one, since the order of precedence settles every pair, and strong robust non-imposition is what excludes it.

The nearest result is that of Nehring and Puppe (2007), on which Puppe and Slinko (2026, Chapter 10) draw. With finitely many voters and a closed Condorcet domain, the monotone rules that satisfy independence and unanimity and take values in the domain are described by winning coalitions, and the anonymous ones are quota rules whose thresholds respect the inclusions between the sets of rankings that put one party above another. There the domain, monotonicity and unanimity are assumed. Here the condition on the family and the natural orientation are conclusions, and the rule need not be monotone to begin with. Nehring and Puppe (2010, Section 4.2) let the values of a monotone rule leave the domain, and they note that no comparably simple characterization of Arrovian rules is known without monotonicity (Nehring and Puppe, 2010, Section 2.4). The same contrast holds with judgment aggregation, where Dietrich and List (2007) characterize quota rules on the universal domain with monotonicity among the axioms: here it is forced with three parties, and not with four.

The escapes. The characterization lists what a mechanism can do to escape. (i) Offer at most two parties: majority rule works (May, 1952). (ii) Offer aligned parties: preferences on the offer are single-peaked, and majority rule with exact ties broken by a fixed order along the line works (Black, 1948). (iii) Guarantee a party with a majority of first choices (with three parties): it is a Condorcet winner. (iv) Constrain second choices so that a quota rule works. Each of (i)–(iii) is a way of forcing the producible shares out of the cyclic orthants; when the producible shares are convex, (iv) is the general form. The next section shows that in the plane, (iv) has a geometric meaning.

6 Separated offers: when parties do solve the problem

6.1 Windows

Take three non-aligned platforms a, b, c. The three bisectors meet at the circumcentre ω and cut the plane into six sectors, one per strict ranking. If ω lies outside the convex hull of the support of μ, this convex hull meets at most four consecutive sectors, and when every party has members these four rankings are those in which one party is never ranked last (Appendix D; this is the "never last" value restriction of Sen, 1966). Call such a set of four rankings a *window*. On a window, a strict majority cycle is impossible.

Definition 3. An offer O is *separated* for μ if no triple of non-aligned platforms of O has its circumcentre in the interior of the convex hull of the support of μ. A mechanism is *separating* if all its offers are separated.

In words, the parties are far apart relative to the electorate. In each triple some party is never ranked last: the members of the two others all rank it second, and only its own members arbitrate freely. Separation thus fixes two of the three arbitrations, the opposite of free arbitration. A separating mechanism cannot react only to sizes and means: if it did, moving members across bisectors while keeping means would produce non-separated offers, so a separating mechanism of that kind that gives every party members offers only aligned platforms (Proposition 14).

6.2 Possibility

Exact ties need a convention. An *arbiter* is a point z_0 of the plane or a direction d, a citizen "at infinity". It ranks the platforms by their distance to z_0, or by their coordinate along d, with a fixed convention for its own indifferences (Definition 5). The rule M_A ranks by majority and breaks exact ties by the arbiter. The arbiter is *on the society's side* if

(A) for every offered non-aligned triple with an exactly tied pair, some open half-plane bounded by a line through the circumcentre carries the whole society and contains the arbiter: z_0 lies in it, or d points into it.

Theorem 3 (Possibility)**.** *Let the mechanism be separating.*

(i) If no offered pair is ever exactly tied, strict majority rule is a strict total order on every offer.

(ii) Under (A), M_A is a strict total order on every offer.

Both rules are anonymous, independent and non-null, and non-imposed as soon as each offered pair is won strictly both ways at some profiles.

Condition (A) holds for the point arbiter z_0 whenever z_0 is interior to the convex hull of every society, for instance when all societies have barycentre z_0 (Corollary 2). The proof is short. On a window, two of the three relations of any cycle hold together in no ranking, so a cycle would need two exact ties decided by the arbiter in the direction of the cycle; and the arbiter's ranking of the triple belongs to the window when the arbiter is on the society's side (Appendix D).

Exact ties are a real obstacle, not a detail. For a fixed triangle of platforms and all societies that separate it, no rule at all satisfies the four axioms, because the window depends on which side of ω the society lies (Proposition 13). So separation alone does not escape; the arbiter does, by tying the convention to the society.

A separating mechanism that offers three non-aligned parties to every society with bounded support, and can give the parties any positive sizes, does exist: place three platforms on a slightly curved arc at the scale of the smallest disc enclosing the society (Proposition 15). Majority rule with ties broken by the direction arbiter $d = -e_1$ works on it; Theorem 3(ii) guarantees no fixed point arbiter, since the half-plane carrying the society moves with the society. The mechanism reacts to the shape of the support, not only to sizes and means, as it must (Proposition 14). The example shows that the impossibility is not about the number of parties as such, but about how the parties are placed relative to the voters.

6.3 Elongated societies

The mean-of-members mechanism is not separating. But for a given society its optimal offer may be separated, and then, with this offer held fixed, majority rule (exact ties broken within the window) is consistent for every redistribution of the citizens over the same support. Consider citizens uniform on a rectangle. When the rectangle is long enough, the optimal offer of k parties is a row of vertical strips, whose platforms are aligned. Boissonnet (2026) shows that this happens if and only if the length exceeds a threshold between $0.56\,k$ and $0.70\,k$ times the width; below the threshold, every optimal offer found numerically is neither aligned nor separated.

What matters is the number of parties. Two parties always escape. With parties at the mean of their members, each further party needs the society to stretch further along one axis, so a system with many such parties makes majority rule consistent only in a nearly one-dimensional society. Other mechanisms, such as the arc above, can do better. Nothing here involves the electoral rule; the next section lets it shape the offer.

7 Formation by instrumental support

The mechanisms so far read the geometry of the society: sizes, means, cells. None reads the electoral rule. Yet Duverger's reason for his law is instrumental: a party that cannot change the outcome loses its supporters. This section models that reason. Proto-parties are born; a party survives if enough citizens have an interest in its existence; the interest depends on who would be elected without it, hence on second choices. This is a lever of formation that acts on second choices without restricting where parties are born. With three births it screens the cyclic configurations when the elected outcome jumps with the votes, at the price of leaving mostly one party; as soon as every party weighs on the expected outcome, it does not. Formation only changes the domain $\mathcal{D}$; the axioms and the reduction are unchanged. Proofs are in Appendix S2.

Proto-parties. Parties are born at positions $z_1, \dots, z_K$. An electoral rule assigns to each set A of survivors a random elected platform Y_A, a party of A or a point of the convex hull of their platforms, whose law depends on the society only through the masses of the rankings of A. Three rules serve as examples: *plurality*, the party with the largest first-choice mass wins; *runoff*, the two largest meet in a duel; the *proportional compromise* $Y_A = \sum_{j \in A} w_j z_j$ with w_j proportional to the first-choice mass of j raised to a power $\gamma \geq 1$. Each may be deterministic or noisy. With noise, independent Gaussian shocks of variance σ^2 hit the first-choice masses, and under runoff also the two shares of the duel; the compromise uses the positive parts of the shocked masses (equal weights if none is positive). If no party survives there is no government, which every citizen ranks below every platform by more than any cost. Losses are quadratic.

Citizen x gains from the existence of $j \in A$

$$G_j(x; A) = \mathbb{E}|x - Y_{A\setminus j}|^2 - \mathbb{E}|x - Y_A|^2,$$

and *supports* j if $G_j(x; A) > \kappa$, where $\kappa \geq 0$ is the cost of support; $S_j(A)$ is the mass of the supporters of j. Fix $\varepsilon \in (0, \frac{1}{2})$. A set A of born parties is *stable* if $S_j(A) \geq \varepsilon$ for every $j \in A$ and $S_j(A \cup \{j\}) < \varepsilon$ for every born $j \notin A$. The offers are the stable sets of at least two parties; a society whose only stable sets are single parties receives no offer. Under plurality or runoff without noise, a society is *free of exact ties* if, in every contest among born parties, the first-choice masses are distinct and no duel is tied; other societies receive no offer under these two rules, as exact ties were set aside in Theorem 3(i). The noisy rules and the compromise are defined at every society. The offers depend on the distribution only, so this is a mechanism in the sense of Section 2 (Assumption 1).

A citizen supports a party for what its existence does to the election, not for its platform. The first proposition says why this matters: survival that reads sizes alone leaves second choices free.

Proposition 2 (Sizes are not enough)**.** *Fix the births and a non-aligned triple T among them, and suppose T is offered at a society if and only if its vector of first-choice masses lies in a set Γ_T. Then arbitration is free on T. If Γ_T is convex and symmetric, or $\Gamma_T = \{m : \min_j m_j \geq \varepsilon\}$ with $\varepsilon \leq \frac{1}{3}$, every valid rule that is non-imposed on the shares of T is null on T.*

In these two cases T is a star triple as soon as every share of a pair of T produced at some offer is also produced with T offered. The proof is the construction of Section 4: the members of c who prefer a to b move across the bisector of a and b without changing anybody's first choice, and sizes do not see them. A threshold on vote shares, or a support that each party draws from its own electorate, reads sizes only. Sizes may still help when Γ_T forces a majority party, as in Section 3.

Plurality without noise. The elected platform is the plurality winner $W(A)$. Party j is *useless* in A if $W(A \setminus j) = W(A)$: nobody gains from its existence, so $S_j(A) = 0$. A party that is not useless is a spoiler. It splits the vote of another party that would otherwise win, and its supporters are the citizens who prefer the actual winner to that party by more than κ.

Proposition 3 (Plurality without noise)**.** *Let the rule be plurality without noise and the society free of exact ties.*

(i) No pair of born parties is stable.

(ii) With three births, the triple is stable if and only if its plurality winner loses both duels and the two spoilers have support at least ε.

(iii) With three births and $\kappa = 0$, a party is stable alone if and only if it is a Condorcet winner.

(iv) With four births, the offer of all four parties can be stable and cyclic, for every ε and every κ small relative to the size of the offer.

By (ii), with three births, majority rule is valid and non-imposed on the offers, and a society whose triple cycles receives no offer. By (iii), whenever the triple is stable, so is its Condorcet winner alone. By (iv), majority rule is not valid with four births. Under runoff without noise, with two or three births, no set of two or more parties is stable (Supplementary Appendix S2). Part (ii) is the escape. In a stable triple the plurality winner owes its victory to two splits: each other party survives because its removal would hand the election to the

third. So the winner loses both duels, and a tournament with a Condorcet loser is transitive. In a cycle, the plurality winner beats one of the two others, which is then useless. Part (iv) breaks this with the plurality winner and three spoilers, each splitting the vote of the next one around a cycle $b \succ c \succ d \succ b$; the society is explicit in the appendix. Under runoff with three births, the party eliminated in the first round never changes who meets whom, so it is useless, and so is the loser of a pair. With four births this fails: removing an eliminated party can change who reaches the duel (the appendix gives a stable offer of four parties).

The escape has a price. No pair is ever stable, so the mechanism never produces a two-party system. And by part (iii) every stable triple comes with a stable single party. In a sample of three thousand societies with uniformly random sector masses and $\kappa = 0$, the Condorcet winner alone is stable in 94 per cent, the triple in about 3 per cent (the figure moves slightly with ε), and the 6 per cent of cyclic societies have no stable set. Instrumental survival screens cycles, but what it leaves is mostly a single party, which is not an offer at all.

Positive weights. The screening relies on a step: without noise, a losing party may change nothing. Under any aggregate uncertainty every party wins with some probability, and under a proportional compromise every party moves the outcome. Then no party is useless.

Theorem 4 (Positive weights)**.** *Let the births be a non-aligned triple T, $\kappa \geq 0$ and $\varepsilon \leq \frac{1}{6}$. Let the rule be plurality or runoff with noise, or the proportional compromise with any $\gamma \geq 1$, with or without noise. Then T is a star triple, and no rule satisfies anonymity, independence, non-imposition and non-nullness. In particular, majority rule is not valid.*

The proof places the citizens of each sector far from the parties, in a direction d. Such a citizen supports i when d points along the shift n_i of the expected outcome caused by i. The directions that do so form a half-circle, and every half-circle contains all the directions in which some party k is preferred to some party l: it reaches a mass of citizens at least s_{kl}, and more unless n_i points exactly along $z_k - z_l$. On the arm of the cross through $(w, \frac{1}{2}, \frac{1}{2})$, every such share is at least one half except $s_{ab} = w$. And the shift due to a never points exactly along $a - b$: the members of a all rank c second, so removing a changes the weight of c. Each party thus finds supporters among half of the citizens, and gets a third of them.

The far placement is not what drives the theorem. When support is free ($\kappa = 0$), a citizen supports i when she lies on one side of a line orthogonal to n_i. Under plurality and runoff the elected platform is always a party, and the three parties are at the same distance from the circumcentre ω of T; so this line passes through ω, and whether a citizen supports a party depends on her direction from ω, not on her distance.

Proposition 4 (Bounded societies)**.** *Keep the assumptions of Theorem 4 and admit only the societies carried by a disc centred at the circumcentre of T. There are $R_0 \geq 0$ and $\nu > 0$, depending only on T, γ and σ, with $R_0 = 0$ under plurality and runoff, such that the conclusion of Theorem 4 holds under the three noisy rules whenever the radius of the disc exceeds $R_0 + \kappa/\nu$, and under the compromise without noise with $\kappa = 0$ whenever it exceeds R_0.*

Under the compromise the line need not pass through ω; without noise it passes through a point of the triangle, so the citizens must be at a distance of order R_T divided by a margin that depends on the angles of T and on γ. Without noise the bound is explicit and holds for every $\gamma \geq 1$: on the arm, each shift n_i stays in a cone spanned by two edges of T, whatever w, with coefficients whose ratio is bounded where the proof uses it (Supplementary Appendix S2). The explicit bound is crude, a few hundred circumradii; in numerical checks on three triangles, a disc of 1.5 circumradii suffices for $\gamma = 1$ without noise, and 8 for $\gamma = 3$ and $\sigma = 0.05$.

Two reservations remain. First, a cost of support has a price in distance. With $\kappa > 0$, ν is the product of the smallest shift of the expected outcome that a party can cause and of an angular margin, and both can be tiny: the shift when a party's chance of winning is tiny, the margin when a shift is nearly parallel to an edge; with $\kappa = 0.02$ and $\sigma = 0.1$, the placement of the proof needs up to $2 \cdot 10^4$ times the size of the triangle, and for $\sigma = 0.05$ it is beyond any meaningful scale. Under the deterministic compromise, the shift of a party with almost no members vanishes, and with $\kappa > 0$ the bounded version is open. Second, the theorem has three births; with more, outsiders add stability conditions that the proof does not control.

The link between the rule and the offer, assumed in Section 6, is now modelled, and the verdict reads the rest of the paper backwards. A rule whose outcome jumps with the votes screens cyclic configurations through the spoilers, by killing almost every party. A rule that lets every party weigh, by chance or by proportion, keeps the parties and brings Arrow back.

8 Robustness

The main theorem was stated in a reference framework. Four variants are treated in the Supplementary Appendix; I record what changes. The Supplementary Appendix also treats offers with collinear triples (Appendix S1.2).

Ties of positive mass. If distributions may charge lines, shares become pairs (s_{ab}, s_{ba}) with $s_{ab} + s_{ba} \leq 1$. The star lemma holds verbatim. Impossibility holds for a fixed non-aligned offer, on the domain of distributions that charge no line or on any domain that contains the mixtures of uniform distributions on discs and segments; for the minimum-size mechanism with $k \geq 3$ when citizens on a boundary belong to no party; for parties at the mean of their members with $k \geq 3$ when all equilibria are admitted and boundary citizens belong to no party (or are split equally); and with $k \geq 4$ under the optimal selection or when offers must put no mass on boundaries. With exactly three mean-of-members parties and boundaries of zero mass, the star method cannot work and the question is open (Appendix S1.1). With ties, free arbitration with minimum sizes ε gives the impossibility only for $\varepsilon \geq \frac{1}{4}$, against every $\varepsilon \leq \frac{1}{3}$ without ties (Corollary S1). Impossibility is not monotone in the domain: adding a single distribution can make a rule possible.

Equilibrium selection. For parties at the mean of their members, any selection that places the parties on k tight, well-separated clusters when the citizens form such clusters gives the impossibility (Proposition 6). A selection that does not can escape: in the minimum-size model, always placing three platforms on the principal axis with equal cells (for societies with a finite second moment) makes majority rule valid; a different selection, three platforms on a horizontal line at averaged quantiles, does the same and is continuous for weak convergence when $\varepsilon < 1/3$ (Appendix S1.3). Continuity is not the right condition; reacting to clusters is. In the minimum-size model, the impossibility under a cluster-recognizing selection is proved under a non-imposition that bears on interior shares; under Wilson's axiom alone it is open.

Crowded offers. If the policy space is bounded and platforms are forced close together, the far-away parties used in the proofs are not available. A weaker criterion, checkable by linear programming for a fixed offer, still gives impossibility when the crosses through the centre reach the extreme shares of each pair; no non-aligned crowded offer admitting a rule was found (Appendix S1.4).

Weak non-imposition in the characterization. Under a weaker robust non-imposition (each pair weakly won both ways on an interval) and non-nullness, Theorem 2 reads: a rule exists if and only if a quota rule or an *edge rule* exists, the latter being indifferent except at extreme shares (Supplementary Appendix S4). Under Wilson's axiom alone the equivalence fails: a rule may be strict everywhere and flip at a single extreme share, with neither a quota rule nor an edge rule available (Example S6).

9 Final remarks

Parties do not solve Arrow's problem by existing. When parties sit at the mean of their members, the second choices of the members stay free, and no anonymous, independent, non-imposed rule can rank the parties consistently; Pareto is not even needed. Parties solve the problem when they constrain those second choices: by bringing the offer down to two options, by aligning it on an axis, or by keeping the parties so far apart that in each triple some party is never ranked last, with exact ties broken by a suitable arbiter. For three parties whose producible shares form a convex set, the quota characterization says exactly how much constraint is enough. Letting the electoral rule decide which parties survive does not change the verdict. Instrumental survival screens cycles only when the outcome jumps with the votes and three parties are born, and then it mostly leaves one party; as soon as every party has a chance or a weight in the outcome, Arrow returns.

The limitation that matters most is the gap between three and more parties in the characterization. For four parties the quota condition fails on sets of shares that a competition produces, because consistency no longer forces a monotone rule to exist. It survives on the domains defined by a family of strict rankings, in the interior and for any number of parties, when every triple spans three dimensions; and on a Condorcet domain without a unanimous pair a quota rule exists on the closed set: majority with ties broken by a ranking of the family. With four parties, every family that carries a rule carries a quota rule. Whether this survives on the families in which a flat triple is a Latin square, for five parties or more, is open; settling it would tell how far the escapes listed by the characterization extend.

The informal claim that two-party systems rescue majority rule is therefore right, and narrower than it sounds. It is not the existence of parties that helps, but the reduction of the offer. Riker himself called the stability of two-party systems "more apparent than real": beneath it, losers keep looking for new issues and new parties that would upset it. The present results add that the reduction to two options is itself a choice that no one made by majority, and they say where that choice has to be made: at the point where a mechanism decides how many parties the voters' geometry will carry.

A Proofs for Sections 2 and 3

This appendix proves the measure-theoretic fact behind the reduction to vote shares, a version of Lemma 1 in which lines may carry mass, and the description of the label triples that come from a weak order. The footnote of Section 2 proves that Pareto is vacuous when platforms are means of members.

Conventions. Σ is the σ-algebra of Lebesgue measurable subsets of $I = [0,1]$, and λ is Lebesgue measure on it. A profile is a map $x : I \to \mathbb{R}^2$ measurable from Σ to the Borel sets of $\mathbb{R}^2$, and $\mu_x = \lambda \circ x^{-1}$. For $a \neq b$ in $\mathbb{R}^2$, let

$$H_{ab} = \{z \in \mathbb{R}^2 : |z-a| < |z-b|\}, \qquad L_{ab} = \{z \in \mathbb{R}^2 : |z-a| = |z-b|\},$$

the open half-plane of points closer to a and the perpendicular bisector of $[a,b]$. For a profile x, let

$$P_{ab}(x) = x^{-1}(H_{ab}), \qquad P_{ba}(x) = x^{-1}(H_{ba}), \qquad T_{ab}(x) = x^{-1}(L_{ab}).$$

These three sets are in Σ and partition I. $P_{ab}(x)$ is the set $\{i : a \succ_i b\}$ of Axiom 2, and $s_{ab}(x) = \lambda(P_{ab}(x)) = \mu_x(H_{ab})$. In the paper μ_x charges no line, so $T_{ab}(x)$ is null and $s_{ba} = 1 - s_{ab}$. The proofs below do not use this assumption: they carry the three-piece partition, and the reduction holds with the pair (s_{ab}, s_{ba}) in place of s_{ab}.

I read "measure-preserving bijection" in Axiom 1 in the usual way: a bijection $\sigma : I \to I$ such that $\sigma^{-1}(E) \in \Sigma$ and $\sigma(E) \in \Sigma$ for every $E \in \Sigma$, and $\lambda(\sigma^{-1}(E)) = \lambda(E)$ for every $E \in \Sigma$.

A.1 Shares as pairs

Lemma 1 assumed that no line carries mass, so that $s_{ba} = 1 - s_{ab}$. Without this assumption the reduction holds with the pair (s_{ab}, s_{ba}) in place of s_{ab}.

Lemma 4. *Let F satisfy anonymity and independence, let $(x, O), (y, O') \in \mathcal{D}$ and $a \neq b$ in $O \cap O'$. If $s_{ab}(x) = s_{ab}(y)$ and $s_{ba}(x) = s_{ba}(y)$, then $F(x, O)$ and $F(y, O')$ rank a and b in the same way.*

Proof. The triples $(P_{ab}(y), P_{ba}(y), T_{ab}(y))$ and $(P_{ab}(x), P_{ba}(x), T_{ab}(x))$ are two partitions of I into sets of Σ with equal masses, the third mass being determined by the two others. The construction of footnote 5 gives a measure-preserving bijection σ of I and a Borel set I_0 of measure 1 such that σ maps each piece of the first partition, within I_0, into the corresponding piece of the second. Let $z = x \circ \sigma$. For a Borel set $E \subset \mathbb{R}^2$, $z^{-1}(E) = \sigma^{-1}(x^{-1}(E))$ is in Σ and has the measure of $x^{-1}(E)$; so z is a profile, $\mu_z = \mu_x$, $(z, O) \in \mathcal{D}$ by Assumption 1, and anonymity gives $F(z, O) = F(x, O)$. A citizen $i \in I_0$ lies in exactly one piece of the first partition, and $\sigma(i)$ lies in the corresponding piece of the second; hence $P_{ab}(z) \cap I_0 = \sigma^{-1}(P_{ab}(x)) \cap I_0 = P_{ab}(y) \cap I_0$, and likewise for P_{ba}. The sets $\{a \succ_i b\}$ and $\{b \succ_i a\}$ thus coincide under z and under y up to the null set $I \setminus I_0$, and independence, applied to (z, O) and (y, O'), gives the conclusion. □

Remark 1 (Independence within an offer). Independence is used once, across the offers O and O'. If it is only required within an offer, the proof gives the conclusion for $O' = O$: the ranking of a and b then depends on the shares of the pair and on the offer, and the data of a pair are not pooled across offers, as the sets D_{ab} of Theorem 1 require.

A.2 Weak orders on a triple

Fix three distinct points a, b, c and the cyclic order $a \to b \to c \to a$. A weak order on $\{a, b, c\}$ gives a triple of labels, one for each of the pairs (a, b), (b, c), (c, a): F if the first point is strictly above the second, S if it is strictly below, I if they are indifferent.

Lemma 5. *A triple of labels comes from a weak order on $\{a, b, c\}$ if and only if all three labels are I, or at least one is F and at least one is S.*

Proof. A weak order on three elements is either total indifference, which gives $(\mathsf{I}, \mathsf{I}, \mathsf{I})$, or has an element strictly above both others or strictly below both others. If a is strictly above b and c, the labels of (a, b) and (c, a) are F and S; if a is strictly below both, they are S and F; the cases of b and c follow by rotating the names. Conversely, let a triple contain an F and an S. If it has no I, it is one of the six triples of $\{\mathsf{F}, \mathsf{S}\}^3$ other than $(\mathsf{F}, \mathsf{F}, \mathsf{F})$ and $(\mathsf{S}, \mathsf{S}, \mathsf{S})$: $(\mathsf{F}, \mathsf{F}, \mathsf{S})$ is $a \succ b \succ c$, and the five others are obtained by rotating the names and by reversing the order, which exchanges F and S. If it has one I, say on (c, a), then $(\mathsf{F}, \mathsf{S}, \mathsf{I})$ is $a \sim c \succ b$ and $(\mathsf{S}, \mathsf{F}, \mathsf{I})$ is $b \succ a \sim c$, and the other positions of the I follow by rotation. □

B Proofs for Section 4

Throughout, a, b, c are three platforms, $T = \{a, b, c\}$, shares are written in cyclic order $v = (s_{ab}, s_{bc}, s_{ca})$, and $h = (1/2, 1/2, 1/2)$. *Majority rule* gives the label F to a share above $1/2$, the

label S to a share below $1/2$, and the label I to the share $1/2$. A size vector $m = (m_a, m_b, m_c)$ lies in the simplex $\Sigma = \{m \in [0,1]^3 : m_a + m_b + m_c = 1\}$, and

$$B(m) = [m_a, 1 - m_b] \times [m_b, 1 - m_c] \times [m_c, 1 - m_a],$$

which is the box of Section 4, since $m_a + m_c = 1 - m_b$, and so on.

B.1 Tools

Three lemmas serve the proofs below. When a, b, c are non-aligned, ω is their circumcentre and R_T the circumradius. The *cell* of a point t of an offer O, relative to O, is the open set $\{x : \ |x - t| < |x - p| \text{ for every } p \in O \setminus \{t\}\}$. For a strict order π on T, let $f(\pi) \in \{0,1\}^3$ record, in cyclic order, whether $a \succ b$, $b \succ c$ and $c \succ a$ under π: it is the share vector of a citizen whose ranking is π. Let

$$\mathcal{P} = \{(p,q,r) \in [0,1]^3 : \ 1 \le p + q + r \le 2\}, \qquad \mathcal{P}^\circ = \{(p,q,r) \in (0,1)^3 : \ 1 < p + q + r < 2\};$$

$\mathcal{P}^\circ$ is the interior of $\mathcal{P}$.

Lemma 6 (Sectors and cells)**.** *Let a, b, c be non-aligned.*

(a) The three perpendicular bisectors meet at ω and cut the plane into six open sectors. Each strict order on T is the ranking by distance on exactly one sector. Two consecutive sectors carry rankings that differ by one exchange of adjacent positions, and two opposite sectors carry reversed rankings. In the angular order around ω, the six rankings thus run once through the cycle abc, acb, cab, cba, bca, bac, in one orientation or the other: the hexagon of the six rankings of Appendix C.4.

(b) The cell of $t \in T$ relative to T is an open convex cone with apex ω. It is the union of the two sectors where t is first and of the open ray r_t from ω that separates them. The ray r_t lies on the bisector of the two other points.

(c) $\mathcal{P}$ is the convex hull of the six vectors $f(\pi)$. The share vectors $\sum_\pi w_\pi f(\pi)$ obtained from six strictly positive masses w_π of sum 1 form exactly $\mathcal{P}^\circ$.

Proof. (a) The bisectors are three distinct lines through ω: two would coincide only if two sides of the triangle were parallel. They cut the plane into six open sectors, and a sector lies on one open side of each bisector. The side of the bisector of $\{x, y\}$ decides whether x or y is closer, so the ranking is strict and constant on a sector. Two sectors with the same three sides lie in one intersection of three open half-planes, which is convex and meets no bisector, so they coincide: distinct sectors carry distinct rankings, and six sectors against six orders make the correspondence one to one. Two consecutive sectors are separated by a half-line of one bisector, say of $\{x, y\}$, which the two other bisectors meet only at ω; so they lie on the same side of those two, and their rankings differ exactly on $\{x, y\}$. Both are strict orders, so x and y are adjacent in both: if z stood between them, reversing x and y alone would leave $x \succ z \succ y$ together with $y \succ x$. The point reflection through ω maps each sector onto the opposite one and changes the side of every bisector, so opposite sectors carry reversed rankings. Finally, each order on three items admits exactly two exchanges of adjacent positions, and the graph they form on the six orders is the cycle $abc, acb, cab, cba, bca, bac$; the angular order of the sectors is a closed walk along its edges through its six vertices, so it runs once around it.

(b) The cell of a is $\{x : |x - a| < |x - b|, \ |x - a| < |x - c|\}$, the intersection of two open half-planes bounded by two distinct lines through ω: an open convex cone with apex ω, of angle less than π. On the bisector of $\{b, c\}$, the function $|x-a|^2 - |x-b|^2$ is affine and vanishes only at ω, where this bisector meets that of $\{a, b\}$; so it has opposite signs on the two open

half-lines from ω. On one of them a is strictly first. This open ray r_a lies in the cell. The rest of the cell consists of the points where a is first and b, c are not tied: the two sectors abc and acb, on the two sides of the line of r_a. The same holds for b and c.

(c) A strict order gives one or two forward relations among $a \succ b$, $b \succ c$, $c \succ a$, never zero or three, which would be a cycle. So the six vectors $f(\pi)$ are the vectors of $\{0,1\}^3$ other than $(0,0,0)$ and $(1,1,1)$, and they lie in $\mathcal{P}$. At a vertex of the polytope $\mathcal{P}$, three linearly independent constraints are active. The constraints $p+q+r=1$ and $p+q+r=2$ are never active together, so at least two coordinates are 0 or 1, and the third is then an integer. The vertices of $\mathcal{P}$ are thus the six vectors $f(\pi)$, and $\mathcal{P}$ is their convex hull. If all $w_\pi > 0$, each coordinate of $\sum_\pi w_\pi f(\pi)$ is a mixture of 0 and 1 with positive weight on both, so it lies in $(0,1)$, and the sum of the coordinates is a mixture of 1 and 2 with positive weight on both, so it lies in $(1,2)$. Conversely, let $v \in \mathcal{P}^\circ$. The centre h is $\sum_\pi \frac{1}{6} f(\pi)$. Since v is interior, $y = h + (v-h)/\theta$ lies in $\mathcal{P}$ for some $\theta \in (0,1)$; write $y = \sum_\pi y_\pi f(\pi)$ with weights $y_\pi \geq 0$ of sum 1. Then $v = (1-\theta)h + \theta y = \sum_\pi \left(\frac{1-\theta}{6} + \theta y_\pi\right) f(\pi)$, with all weights strictly positive. □

Lemma 7 (Two discs with a prescribed mean)**.** *Let a, b, c be non-aligned, $t \in T$, K the cell of t relative to T, and K_1, K_2 its two sectors.*

(a) The barycentre of a distribution with a finite first moment carried by an open convex set lies in that set. In particular, a distribution that gives K a positive mass has its mean on K inside K.

(b) $\theta(K_1 - \omega) + (1-\theta)(K_2 - \omega) = K - \omega$ for every $\theta \in (0,1)$. Hence, for $g \in K$, $m > 0$ and $\theta \in (0,1)$, there are $c_1 \in K_1$ and $c_2 \in K_2$ with $\theta c_1 + (1-\theta)c_2 = g$, and a radius $\rho > 0$ with $\bar{B}(c_1, \rho) \subset K_1$ and $\bar{B}(c_2, \rho) \subset K_2$. The distribution that puts the mass θm uniformly on the first disc and the mass $(1-\theta)m$ uniformly on the second is carried by K, charges no line, has mass m and mean g.

Proof. (a) Let C be open and convex, ν a non-zero finite measure carried by C with a finite first moment, and g its barycentre. If $g \notin C$, some $u \neq 0$ satisfies $\langle u, y\rangle \leq \langle u, g\rangle$ for every $y \in C$, and the inequality is strict: at a point $y_0 \in C$ of equality, $y_0 + su$ would lie in C for small $s > 0$ and violate it. Integrating against $\nu/\nu(C)$ gives $\langle u, g\rangle < \langle u, g\rangle$. So $g \in C$. The cell K is open and convex by Lemma 6(b).

(b) Translate so that $\omega = 0$. Then K_1, K_2 and K are open convex cones, so $sK = K$ for $s > 0$, and likewise for K_1 and K_2. Hence $\theta K_1 = K_1$ and $(1-\theta)K_2 = K_2$, and the claim is $K_1 + K_2 = K$. If $x \in K_1$ and $y \in K_2$, then $x + y = 2 \cdot \frac{x+y}{2} \in 2K = K$, since K is convex and contains $K_1 \cup K_2$. Conversely, let $z \in K$ and let ℓ be the line of r_t. By Lemma 6(b), K_1 and K_2 are the parts of K on the two sides of ℓ, and $K \cap \ell = r_t$. If $z \in K_1$, pick $y \in K_2$ so small that $z - y \in K_1$, which is open; then $z = (z-y) + y$. The case $z \in K_2$ is symmetric. If $z \in r_t$, pick $y \in K_2$ so small that $z - y \in K$, which is open. As $z \in \ell$ and y lies strictly on the side of K_2, the point $z - y$ lies strictly on the side of K_1, so $z - y \in K_1$. Hence $K \subset K_1 + K_2$.

Translating back, $g - \omega \in K - \omega$ gives c_1 and c_2. The sectors are open, which gives ρ. A uniform distribution on a disc charges no line and has the centre as mean. The mixture has mass $\theta m + (1-\theta)m = m$ and mean $\theta c_1 + (1-\theta)c_2 = g$. □

Lemma 8 (Far parties)**.** *Let a, b, c be non-aligned, and write K_t for the cell of $t \in T$ relative to T. Let $R > 0$, $n \geq 0$, and let $v_1, \ldots, v_n$ be distinct vectors such that, for each j, $\omega + v_j$ lies in the cell K_{t_j} of some $t_j \in T$. For $\tau > 0$ let $d_j = \omega + \tau v_j$ and $O_\tau = T \cup \{d_1, \ldots, d_n\}$. There is τ_0, which depends on T, R and the v_j only, such that for every $\tau > \tau_0$:*

(i) the $n+3$ points of O_τ are distinct;

(ii) for $t \in T$, $\bar{B}(\omega, R) \cap K_t$ lies in the cell of t relative to O_τ; for each j, $\bar{B}(d_j, 1)$ lies in the cell of d_j relative to O_τ;

(iii) $\bar{B}(d_j,1) \subset K_{t_j}$, *so its points rank* t_j *first on* T. *If* $\omega + v_j$ *lies in a sector of* t_j, *so does* $\bar{B}(d_j,1)$. *If* $\omega + v_j \in r_{t_j}$, *the line of* r_{t_j} *cuts* $\bar{B}(d_j,1)$ *into two half-discs, one in each sector of* t_j.

Consequently, let ν *be a probability distribution carried by* $\bar{B}(\omega,R)$ *that charges no line, let* $\eta_1,\dots,\eta_n > 0$ *with* $\delta = \sum_j \eta_j < 1$, *and let* $\lambda_1,\dots,\lambda_n$ *be probability distributions carried by* $\bar{B}(0,1)$ *that charge no line. For* $\tau > \tau_0$, *the distribution*

$$\mu = (1-\delta)\,\nu + \sum_{j=1}^{n} \eta_j\,(d_j + \lambda_j),$$

where $d_j + \lambda_j$ *is the translate of* λ_j *by* d_j, *charges no line. The cell of* $t \in T$ *relative to* O_τ *carries* $(1-\delta)$ *times the restriction of* ν *to* K_t: *its mass is* $(1-\delta)\nu(K_t)$ *and, when this is positive, its mean is the mean of* ν *on* K_t. *The cell of* d_j *carries* $\eta_j(d_j+\lambda_j)$: *its mass is* η_j *and its mean is* d_j *plus the mean of* λ_j, *which is* d_j *when* λ_j *is uniform on a disc centred at* 0.

Proof. If $n = 0$ there is nothing to prove; let $n \geq 1$. Let $s = \min_j |v_j| > 0$, let $\varepsilon_0 = \min_{i\neq j} |v_i - v_j| > 0$ if $n \geq 2$ and $\varepsilon_0 = 1$ otherwise, and let $\rho_j > 0$ be the distance from $\omega + v_j$ to the complement of K_{t_j}, or to the complement of its sector when $\omega + v_j$ lies in a sector; these sets are open. The map $x \mapsto \omega + \tau(x-\omega)$ preserves K_{t_j} and its sectors, which are cones with apex ω, and multiplies distances by τ; so d_j is at distance $\tau\rho_j$ from that complement. Let

$$\tau_0 = \max\Big(\frac{2R+R_T}{s},\ \frac{2+R_T}{s},\ \frac{2}{\varepsilon_0},\ \max_j \frac{1}{\rho_j}\Big),$$

and let $\tau > \tau_0$.

(i) $|d_j - \omega| = \tau|v_j| \geq \tau s > R_T = |t-\omega|$ for $t \in T$, so $d_j \notin T$; and the d_j are distinct since the v_j are.

(ii) Let $x \in \bar{B}(\omega,R) \cap K_t$. Then $|x-t| \leq |x-\omega| + |\omega - t| \leq R + R_T$, while $|x-d_j| \geq |d_j-\omega| - |x-\omega| \geq \tau s - R > R + R_T$. So x is strictly closer to t than to every d_j, and than to the two other points of T since $x \in K_t$. Let $x \in \bar{B}(d_j,1)$. For $i \neq j$, $|x-d_i| \geq |d_i - d_j| - 1 = \tau|v_i - v_j| - 1 \geq \tau\varepsilon_0 - 1 > 1 \geq |x-d_j|$; and for $t \in T$, $|x-t| \geq |d_j-\omega| - |\omega - t| - |x-d_j| \geq \tau s - R_T - 1 > 1 \geq |x-d_j|$.

(iii) The distance from d_j to the complement of K_{t_j}, or of its sector, is $\tau\rho_j > 1$, so $\bar{B}(d_j,1)$ lies in it. If $\omega + v_j \in r_{t_j}$, then d_j lies on the line ℓ of r_{t_j}, and ℓ cuts the disc, centred on ℓ, into two half-discs. They lie in K_{t_j} on the two sides of ℓ, that is, in the two sectors of t_j (Lemma 6(b)).

The consequences. μ has mass 1 and charges no line, since ν and the λ_j do not. The cell of t relative to O_τ lies in K_t and, by (ii), contains $\bar{B}(\omega,R) \cap K_t$; as ν is carried by $\bar{B}(\omega,R)$, the restrictions of ν to this cell and to K_t coincide. The far discs lie in the cells of the d_j by (ii), hence outside the cell of t. So the cell of t carries $(1-\delta)\nu|_{K_t}$. The cell of d_j contains $\bar{B}(d_j,1)$, which carries $\eta_j(d_j+\lambda_j)$; it meets no other far disc, and it meets $\bar{B}(\omega,R)$ in a ν-null set, since a point of $\bar{B}(\omega,R)$ lies on a bisector of T or in some K_t, hence in the cell of t. So the cell of d_j carries $\eta_j(d_j+\lambda_j)$, whose mean is d_j plus the mean of λ_j. □

B.2 Free arbitration on one triple

This subsection applies the recipe of Section 3 when one triple is offered. Arbitration is free at every producible size vector, and $M \subset \Sigma$ is the set of producible size vectors. The producible share vectors form $V = \bigcup_{m\in M} B(m)$, and non-imposition is read on $D_i = \pi_i(V)$: no label is constantly F or constantly S on D_i. The box $B(m)$ contains h if and only if $m_a \leq 1/2 \leq 1-m_b$, $m_b \leq 1/2 \leq 1-m_c$ and $m_c \leq 1/2 \leq 1-m_a$, that is, if and only if $\max m \leq 1/2$. Consider the condition

$$\text{(H)} \qquad \pi_i(V) = \bigcup\{\pi_i(B(m)) :\ m \in M,\ \max m \leq 1/2\} \quad \text{for } i = 1,2,3:$$

every share is already produced at sizes without a majority party. Under (H), the boxes $B(m)$ with $\max m \leq 1/2$ are products contained in V, they all contain h, and their factors cover the projections, so Lemma 3 applies.

Corollary 1 (Free arbitration on one triple)**.** *Every valid, non-imposed rule is null on the triple in each of the following cases.*

(i) *M is non-empty, convex, and invariant under the permutations of a, b, c. Neither a non-empty interior nor closedness is needed.*

(ii) *Minimum sizes: $M = M_\varepsilon = \{m \in \Sigma : \ m_a, m_b, m_c \geq \varepsilon\}$ with $0 \leq \varepsilon \leq 1/3$.*

(iii) *$M = \{m^0\}$ is a single point.*

In cases (i) and (ii), (H) holds. In case (iii), (H) fails if $\max m^0 > 1/2$.

Proof. (i) I check (H) for the first coordinate. The inclusion $\supseteq$ is clear. Let $w \in \pi_1(V)$. There is $m^* \in M$ with $m^*_a \leq w \leq 1 - m^*_b$. Let τ_{bc} exchange the sizes of b and c, and τ_{ac} those of a and c. By symmetry $\tau m^* \in M$, and by convexity the midpoint $(m^* + \tau m^*)/2$ is in M.

- If $w \leq 1/2$, let $m' = (m^* + \tau_{bc} m^*)/2 = (m^*_a, \frac{1-m^*_a}{2}, \frac{1-m^*_a}{2})$. Since $m^*_a \leq w \leq 1/2$, $\max m' \leq 1/2$. The first factor of $B(m')$ is $[m^*_a, \frac{1+m^*_a}{2}]$, and it contains w, because $w \leq 1/2 \leq \frac{1+m^*_a}{2}$.

- If $w \geq 1/2$, let $m' = (m^* + \tau_{ac} m^*)/2 = (\frac{1-m^*_b}{2}, m^*_b, \frac{1-m^*_b}{2})$. Since $m^*_b \leq 1 - w \leq 1/2$, $\max m' \leq 1/2$. The first factor of $B(m')$ is $[\frac{1-m^*_b}{2}, 1 - m^*_b]$, and it contains w, because $\frac{1-m^*_b}{2} \leq 1/2 \leq w$.

M is invariant under the rotation $a \to b \to c \to a$, which maps the first coordinate to the other two. So (H) holds for all three coordinates.

(ii) M_ε contains $(1/3, 1/3, 1/3)$, so it is non-empty for $\varepsilon \leq 1/3$. It is convex and symmetric, and (i) applies.

In both cases, Lemma 3 gives the conclusion, as explained before the corollary. (iii) If $M = \{m^0\}$, then $V = B(m^0)$ is a product. Lemma 3 applies to the family made of this single product and to any point $h' \in B(m^0)$: the factors of the product are the projections of V. Hypothesis (H) is not used, and it fails when $\max m^0 > 1/2$, since then no size vector of M is free of a majority party. □

Symmetry alone is not enough. The reason is that a party with a majority of first choices is a Condorcet winner.

Proposition 5 (A majority party restores majority rule)**.** *Suppose that, for every $m \in M$, some party has strictly more than half of the members. Then majority rule is valid on V.*

Proof. Let $m \in M$ with $m_a > 1/2$. At every point of $B(m)$, $s_{ab} \geq m_a > 1/2$, so the label of (a, b) is F and $a \succ b$; and $s_{ca} \leq 1 - m_a < 1/2$, so the label of (c, a) is S and $a \succ c$. The labels contain an F and an S, so they come from a weak order. The cases $m_b > 1/2$ and $m_c > 1/2$ are the same up to rotation. □

For the permutations of $(0.8, 0.1, 0.1)$, each share ranges over $[0.1, 0.9]$, so majority rule is also non-imposed and non-null.

B.3 Richness for parties at the mean of their members

I read Example 1 with distributions that have a finite first moment, so that the mean of each cell exists, and a finite second moment when the optimal offer is selected. For a distribution μ, $\mathrm{Eq}(\mu)$ is the set of equilibrium offers: sets of k distinct points whose cells have positive mass and have their own platform as mean.

Proposition 6 (Parties at the mean of their members)**.** *In Example 1, with $k \geq 3$ and all equilibrium offers admitted, every offered pair belongs to a star triple. The same holds if only the optimal offer is admitted, and more generally under any selection that places the parties on the clusters when the citizens form k tight, well-separated clusters.*

Lemma 9 (Richness, three parties)**.** *Let a, b, c be non-aligned and $(p, q, r) \in \mathcal{P}^\circ$. There is a distribution μ with compact support, uniform on finitely many discs, such that $T \in \mathrm{Eq}(\mu)$ with $k = 3$ and $(s_{ab}, s_{bc}, s_{ca}) = (p, q, r)$.*

Proof. By Lemma 6(c), there are six strictly positive sector masses w_π of sum 1 whose share vector is (p, q, r). Fix $t \in T$, let π_1, π_2 be the two orders that rank t first, K_1, K_2 their sectors, and $m_i = w_{\pi_i}$. The platform t lies in its own cell K. Lemma 7(b), with $g = t$, $m = m_1 + m_2$ and $\theta = m_1/(m_1 + m_2)$, gives two small discs, one in K_1 carrying the mass m_1 and one in K_2 carrying the mass m_2, whose mixture has mean t. Do the same for the two other platforms, and let μ be the sum of the six uniform pieces. It is a probability distribution with compact support, and it charges no line. The cell of t relative to T contains exactly the two discs of t, so it has the positive mass $m_1 + m_2$ and the mean t. Hence $T \in \mathrm{Eq}(\mu)$. Each sector carries the mass w_π of its order, so the shares are $\sum_\pi w_\pi f(\pi) = (p, q, r)$. □

Lemma 10 (Richness, k parties)**.** *Let $k \geq 3$, let a, b, c be non-aligned and $(p, q, r) \in \mathcal{P}^\circ$. There is a distribution μ with compact support, uniform on finitely many discs, and an offer $O \in \mathrm{Eq}(\mu)$ with $T \subset O$ and $(s_{ab}, s_{bc}, s_{ca}) = (p, q, r)$.*

Proof. For $k = 3$ this is Lemma 9. Let $k > 3$ and $n = k - 3$. Choose n distinct unit vectors $e_1, \ldots, e_n$ such that each $\omega + e_j$ lies in a sector of T, let π_j be the order of that sector, and let $u = \frac{1}{n}\sum_j f(\pi_j)$. For $\eta > 0$, let $\delta = n\eta$ and $(p', q', r') = \big((p, q, r) - \delta u\big)/(1 - \delta)$. As $\eta \to 0$, this point tends to (p, q, r), which is interior to $\mathcal{P}$; fix η with $\delta < 1$ and $(p', q', r') \in \mathcal{P}^\circ$. Let ν be the distribution of Lemma 9 for the target (p', q', r'), and R such that ν is carried by $\bar{B}(\omega, R)$. Apply Lemma 8 with $v_j = e_j$, $\eta_j = \eta$ and λ_j uniform on $\bar{B}(0, 1)$, for some $\tau > \tau_0$; let $O = O_\tau$ and let μ be the resulting distribution. It has compact support and is uniform on finitely many discs. The k points of O are distinct. The cell of $t \in T$ relative to O has mass $(1 - \delta)\nu(K_t) > 0$ and the mean of ν on K_t, which is t since $T \in \mathrm{Eq}(\nu)$; the cell of d_j has mass η and mean d_j. So $O \in \mathrm{Eq}(\mu)$. Shares are linear in the distribution. The near part contributes $(1 - \delta)(p', q', r')$, and the disc of d_j, which lies in the sector of π_j by Lemma 8(iii), contributes $\eta f(\pi_j)$. The shares are $(1 - \delta)(p', q', r') + \delta u = (p, q, r)$. □

Proof of Proposition 6, all equilibria admitted. Let $\{a, b\}$ be an offered pair, c a point off the line (ab), and $T = \{a, b, c\}$. By Lemma 10, $V_T \supseteq \mathcal{P}^\circ$; in particular T is offered.

Shares to cover. In every equilibrium offer containing a and b, the cell of a has positive mass and its members prefer a to b, and the cell of b has positive mass and its members prefer b to a. So $D_{ab} \subset (0, 1)$, and likewise $D_{bc}, D_{ca} \subset (0, 1)$.

Products. For m in the open simplex, let $B^\circ(m) = (m_a, 1-m_b) \times (m_b, 1-m_c) \times (m_c, 1-m_a)$. Its coordinates lie in $(0, 1)$, and their sum lies strictly between $m_a + m_b + m_c = 1$ and $(1 - m_b) + (1 - m_c) + (1 - m_a) = 2$. So $B^\circ(m) \subset \mathcal{P}^\circ \subset V_T$. If $\max m < 1/2$, then $h \in B^\circ(m)$.

Covering. Let $w \in (0, 1/2]$ and $m = (\frac{w}{2}, \frac{2-w}{4}, \frac{2-w}{4})$. Then $\max m < 1/2$, and $w \in (m_a, 1 - m_b) = (\frac{w}{2}, \frac{2+w}{4})$, since $\frac{2+w}{4} > \frac{1}{2} \geq w$. Let $w \in [1/2, 1)$ and $m = (\frac{1+w}{4}, \frac{1-w}{2}, \frac{1+w}{4})$.

Then $\max m < 1/2$, and $m_a < \frac{1}{2} \le w < 1 - m_b = \frac{1+w}{2}$. So the first factors cover $(0,1) \supseteq D_{ab}$. The two other pairs follow by rotation. The open boxes contain h, lie in V_T, and cover D_{ab}, D_{bc}, D_{ca}: T is a star triple. □

The constraint that each party has members who prefer it removes only the boundary of $\mathcal{P}$, and the proof uses only the interior. The second half of Proposition 6, on the optimal selection and on selections that recognize clusters, is proved in the Supplementary Appendix (Appendix S3), with the third point chosen so that the triangle is equilateral, and clusters tight enough that the optimal set is unique.

B.4 Offers that react to sizes and means

Proof of Proposition 1. The open boxes. Let $(x,T) \in \mathcal{D}$ with cells C_a, C_b, C_c of masses $m_a, m_b, m_c > 0$, and let g_t be the mean of μ_x on C_t. By Lemma 7(a), $g_t \in C_t$. Let $\alpha_t \in (0, m_t)$ for each t. Fix t, let K_1, K_2 be its two sectors, K_1 the one whose citizens rank second the party that α_t counts, and $\theta = \alpha_t/m_t$. Lemma 7(b), with $g = g_t$ and $m = m_t$, gives two small discs, one in K_1 carrying the mass α_t and one in K_2 carrying the mass $m_t - \alpha_t$, whose mixture has mean g_t. Doing this for the three parties gives a distribution with compact support, hence a finite first moment, charging no line, whose cells relative to T have the masses m_t and the means g_t. By (S) and the requirement that offers depend on the distribution only, a profile y with this distribution has $(y,T) \in \mathcal{D}$, and its shares are $(m_a + \alpha_c,\ m_b + \alpha_a,\ m_c + \alpha_b)$. As α ranges over the open box $\prod_t (0, m_t)$, these shares fill $B^\circ(m)$.

The star. Every share vector produced with T offered at a society of size vector m lies in $B(m)$. Let w be a share of (a,b) produced at some offer; by hypothesis $\mu_T < w < 1 - \mu_T$. If $w \le \frac{1}{2}$, pick $m \in M_T$ with $m_a < w$ and let $m' = (m_a, \frac{1-m_a}{2}, \frac{1-m_a}{2})$, the midpoint of m and of m with the sizes of b and c exchanged; $m' \in M_T$ by convexity and symmetry. Since $0 < m_a < \frac{1}{2}$, $\max m' < \frac{1}{2}$, and the first factor of $B^\circ(m')$ is $(m_a, \frac{1+m_a}{2}) \ni w$. If $w \ge \frac{1}{2}$, pick $m \in M_T$ with $m_b < 1 - w$ (by symmetry, $\inf\{m_b : m \in M_T\} = \mu_T$) and let $m' = (\frac{1-m_b}{2}, m_b, \frac{1-m_b}{2})$; then $\max m' < \frac{1}{2}$ and the first factor of $B^\circ(m')$ is $(\frac{1-m_b}{2}, 1 - m_b) \ni w$. The boxes $B^\circ(m')$ with $\max m' < \frac{1}{2}$ contain h, so they are products in V_T through h whose first factors cover D_{ab}; the other pairs follow by rotation. T is a star triple. □

B.5 Minimum size

In Example 2, (x,O) is in the domain when μ_x charges no line, O is any set of $k \ge 3$ distinct points, and each cell of O has mass at least ε, with $0 < \varepsilon \le 1/k$.

Proposition 7 (Minimum size)**.** *In Example 2, with $k \ge 3$ and $\varepsilon \le 1/k$, every offered pair belongs to a star triple.*

Proof of Proposition 7. Let $\{a,b\}$ be an offered pair, c a point off the line (ab), $T = \{a,b,c\}$ and ω its circumcentre. For $t \in T$, let r_t be the open ray of Lemma 6(b), which splits the cell of t relative to T into its two sectors, and let u_t be its unit vector. The members of t are the citizens who rank t first on T.

Step 1: shares to cover. In every offer $O \ni a, b$, the cell of a, of mass at least ε, prefers a to b, and the cell of b prefers b to a. So $D_{ab} \subset [\varepsilon, 1-\varepsilon]$, and likewise for the two other pairs.

Step 2: sizes. Take a as the "small party". Let $j_c = \lceil (k-3)/2 \rceil$ and $j_b = \lfloor (k-3)/2 \rfloor$. I look for $m = (\varepsilon, m_b, m_c)$ with $m_b + m_c = 1 - \varepsilon$, $m_b \ge (1 + j_b)\varepsilon$, $m_c \ge (1 + j_c)\varepsilon$ and $m_b, m_c \le 1/2$. This amounts to taking m_c in

$$J = \Big[\max\big((1 + j_c)\varepsilon,\ \tfrac{1}{2} - \varepsilon\big),\ \min\big(\tfrac{1}{2},\ 1 - (2 + j_b)\varepsilon\big) \Big].$$

The interval J is non-empty. First, $1 + j_c = \lceil (k-1)/2 \rceil \le k/2$, so $(1+j_c)\varepsilon \le k\varepsilon/2 \le 1/2$. Second, $(1+j_c)\varepsilon \le 1 - (2+j_b)\varepsilon$ is equivalent to $k\varepsilon \le 1$, since $j_b + j_c = k-3$. Third, $\frac{1}{2} - \varepsilon \le 1 - (2+j_b)\varepsilon$ is equivalent to $(1+j_b)\varepsilon \le \frac{1}{2}$, which holds because $j_b \le j_c$. Fourth, $\frac{1}{2} - \varepsilon \le \frac{1}{2}$. Fix such an m, and let $n_a = \varepsilon$, $n_b = m_b - j_b\varepsilon \ge \varepsilon$ and $n_c = m_c - j_c\varepsilon \ge \varepsilon$. Their sum is $1-\delta$, with $\delta = (k-3)\varepsilon \le 1 - 3\varepsilon < 1$.

Step 3: the offer and the society. For each $t \in T$ fix two small closed discs, one in each sector of t, and call *first* the disc in the sector whose citizens rank second the party counted by α_t: the sector *abc* for a, *bca* for b, *cab* for c. Let R be such that the six discs lie in $\bar{B}(\omega, R)$. For $\beta \in [0,1]^3$, let ν^β be the probability distribution that puts, for each t, the mass $\beta_t n_t/(1-\delta)$ uniformly on the first disc of t and $(1-\beta_t)n_t/(1-\delta)$ uniformly on the second. Let $v_j = j\,u_c$ for $j = 1, \dots, j_c$ and $v_{j_c+i} = i\,u_b$ for $i = 1, \dots, j_b$. These $k-3$ vectors are distinct, since r_b and r_c lie on two distinct lines through ω, and $\omega + v_j$ lies on r_c or on r_b, inside the cell of c or of b relative to T. For $\gamma \in [0,1]^{k-3}$ and a far point d_j on r_t, let λ_j^γ put the mass γ_j uniformly on one of the two halves of $\bar{B}(0,1)$ cut by the line $\mathbb{R}u_t$, and $1-\gamma_j$ uniformly on the other. Apply Lemma 8 with these T, R and v_j and the masses $\eta_j = \varepsilon$, and fix $\tau > \tau_0$; τ_0 depends neither on β nor on γ. Let $O = O_\tau$, which has k points, several of them aligned with ω; this is allowed, since every set of k distinct points is an offer. Let

$$\mu^{\beta,\gamma} = (1-\delta)\,\nu^\beta + \varepsilon \sum_{j=1}^{k-3} (d_j + \lambda_j^\gamma).$$

By the lemma, $\mu^{\beta,\gamma}$ charges no line, the cell of $t \in T$ relative to O has mass $n_t \ge \varepsilon$, and the cell of d_j has mass ε. So (x, O) is in the domain for every profile x with distribution $\mu^{\beta,\gamma}$.

Step 4: sizes and shares relative to T. By Lemma 8(iii), the disc of a far point on r_c lies in the cell of c relative to T, and its two half-discs lie in the two sectors of c; name the half in the sector *cab* the first half, so that γ_j is its mass. Likewise on r_b, with the sector *bca*. The members of a, b and c have masses $n_a = m_a$, $n_b + j_b\varepsilon = m_b$ and $n_c + j_c\varepsilon = m_c$. The members of c who prefer a to b are the first disc of c and the first halves on r_c, of mass $\alpha_c = \beta_c n_c + \varepsilon \sum_{j \le j_c} \gamma_j$, which takes every value in $[0, n_c + j_c\varepsilon] = [0, m_c]$. Likewise $\alpha_b = \beta_b n_b + \varepsilon \sum_{j > j_c} \gamma_j$ takes every value in $[0, m_b]$, and $\alpha_a = \beta_a \varepsilon$ every value in $[0, m_a]$, independently of the others. The whole box $B(m)$ is produced.

Step 5: covering. Since $\max m \le 1/2$, $h \in B(m)$, and the first factor $[\varepsilon, 1-m_b]$ contains $[\varepsilon, 1/2]$. Exchanging the roles of a and b, with b as the small party and the far points on r_a and r_c, gives a produced box that contains h and whose first factor $[m_a, 1-\varepsilon]$ contains $[1/2, 1-\varepsilon]$. By rotation, with small parties b and c for the pair (b,c), and c and a for the pair (c,a), the boxes cover $[\varepsilon, 1-\varepsilon]$ in each coordinate. With Step 1, $D_{ab} = D_{bc} = D_{ca} = [\varepsilon, 1-\varepsilon]$, and all these boxes contain h. So T is a star triple. □

For $\varepsilon = 1/k$ and k even, $J = \{1/2\}$, and the box contains h as a boundary point. This is enough, since the boxes are closed. For $k = 5$ and $\varepsilon = 0.1$, $j_b = j_c = 1$ and $J = [0.4, 0.5]$; with $m = (0.1, 0.45, 0.45)$, one far point on r_b and one on r_c, each of mass 0.1, the box is $[0.1, 0.55] \times [0.45, 0.55] \times [0.45, 0.9]$; it contains h, and its first factor contains $[0.1, 0.5]$.

C Proofs for Section 5

C.1 Setting

Fix an offered triple $\{a, b, c\}$ and write share vectors in cyclic order, $v = (v_1, v_2, v_3) = (s_{ab}, s_{bc}, s_{ca})$. Let V be the set of producible share vectors. I assume V convex and non-empty. Let $D_i = \pi_i(V)$ be its projection on coordinate i; it is an interval. Nothing below uses

$V \subset [0,1]^3$, nor that V is closed or bounded, so I state the results for any convex non-empty $V \subset \mathbb{R}^3$.

A rule on the triple is a triple of labellings $\ell_i : D_i \to \{\mathsf{F}, \mathsf{S}, \mathsf{I}\}$. For coordinate 1, F means $a \succ b$, S means $b \succ a$, and I means $a \sim b$; for coordinate 2, F means $b \succ c$; for coordinate 3, F means $c \succ a$. The *pattern* at v is $(\ell_1(v_1), \ell_2(v_2), \ell_3(v_3))$. It comes from a weak order if and only if it is $(\mathsf{I}, \mathsf{I}, \mathsf{I})$ or contains at least one F and at least one S (Section 3); I call these thirteen patterns *admissible*, and the rule is *valid at* v if its pattern there is admissible. Admissibility depends only on how many F and S a pattern contains.

In this notation, strong robust non-imposition (Axiom 5) says that for each i the sets $\{\ell_i = \mathsf{F}\}$ and $\{\ell_i = \mathsf{S}\}$ both contain a non-degenerate interval of D_i.

In this notation a quota rule picks orientations $\sigma \in \{+1, -1\}^3$, thresholds $t_i \in \operatorname{int} D_i$, and for each coordinate a tie convention: either $F_i = \{u \in D_i : \sigma_i(u - t_i) \geq 0\}$ or $F_i = \{u \in D_i : \sigma_i(u - t_i) > 0\}$. Let $S_i = D_i \setminus F_i$. The quota rule labels F on F_i and S on S_i. It is valid on V if and only if V meets neither $F_1 \times F_2 \times F_3$ nor $S_1 \times S_2 \times S_3$, since a pattern without I is admissible exactly when it is neither $(\mathsf{F}, \mathsf{F}, \mathsf{F})$ nor $(\mathsf{S}, \mathsf{S}, \mathsf{S})$.

Proposition 8 (Quota rules are rules)**.** *A valid quota rule is valid at every point of V and satisfies strong robust non-imposition.*

Proof. Its patterns contain only F and S, and V avoids the two products where the pattern is $(\mathsf{F}, \mathsf{F}, \mathsf{F})$ or $(\mathsf{S}, \mathsf{S}, \mathsf{S})$. So every pattern on V is admissible. Since t_i is interior to D_i, each of F_i and S_i contains a non-degenerate interval, between t_i and an endpoint of D_i. □

C.2 The characterization

Theorem 5. *Let $V \subset \mathbb{R}^3$ be convex and non-empty, with $D_i = \pi_i(V)$.*

(a) If V has empty interior, a valid quota rule exists if and only if no D_i is a single point.

(b) If V has non-empty interior and some rule valid at every point of $\operatorname{int} V$ *satisfies strong robust non-imposition, there is a quota rule valid on all of V, boundary included.*

Proof of Theorem 2. A valid quota rule is a valid rule with strong robust non-imposition (Proposition 8). Conversely, let ℓ be valid on V with strong robust non-imposition. If V has empty interior, each D_i contains a non-degenerate interval, so no D_i is a point, and (a) gives a quota rule. Otherwise ℓ is valid on $\operatorname{int} V$, and (b) gives one. □

Empty interior

Proof of Theorem 5(a). If some D_i is a single point, it has empty interior and no quota rule exists. Suppose that no D_i is a point. A convex set with empty interior lies in a plane $\{v : n \cdot v = c\}$ with $n \neq 0$. If n had a single non-zero component n_i, then $v_i = c/n_i$ on V and D_i would be a point. So n has at least two non-zero components. Fix an index j with $n_j \neq 0$.

Take t in the relative interior of V, which is non-empty since V is convex and non-empty. Coordinate i is not constant on V, so the projection π_i maps the relative interior of V onto the interior of D_i (Rockafellar, 1970, Theorem 6.6). Hence $t_i \in \operatorname{int} D_i$.

Let σ_i be the sign of n_i when $n_i \neq 0$, and $\sigma_i = +1$ otherwise. Take the convention "$>$" on coordinate j and "$\geq$" on the two others. For $v \in V$, since $t \in V$, $n \cdot (v - t) = 0$, that is,

$$\sum_{i:\, n_i \neq 0} |n_i|\, \sigma_i(v_i - t_i) = 0.$$

If $v \in F_1 \times F_2 \times F_3$, all terms are non-negative and the term of index j is positive, so the sum is positive: impossible. If $v \in S_1 \times S_2 \times S_3$, all terms are non-positive, and those of index $i \neq j$ with $n_i \neq 0$ are negative; there is at least one. So the sum is negative: impossible. The quota rule is valid. □

The proof uses only $n \cdot t = c$ and $t_i \in \operatorname{int} D_i$: any such point of the plane serves as threshold. On the plane $v_1 + v_2 + v_3 = 2$ of three cyclically unanimous arbitrating groups, $n = (1, 1, 1)$: the deviations $v_i - t_i$ sum to zero, so they cannot all be positive or all negative, and the convention settles the one case of equality.

Non-empty interior: notation

From here on V has non-empty interior $U = \operatorname{int} V$, an open convex non-empty set, and ℓ is as in Theorem 5(b): valid at every point of U, with strong robust non-imposition. Let $E_i = \pi_i(U)$. It is an open interval, equal to $\operatorname{int} D_i$ by the same theorem of Rockafellar (1970), so the interior values of the shares are exactly the values taken in U. For $u \in E_i$, the *fibre* $U \cap \{v_i = u\}$ is a non-empty open convex subset of the plane $\{v_i = u\}$. For $j \neq i$, let $P_j^i(u) = \pi_j(U \cap \{v_i = u\})$; it is an open interval contained in E_j.

Two symmetries shorten the proof. Permuting the coordinates, or replacing a coordinate v_i by $2t_i - v_i$ (a reflection, with labels carried along: $\ell_i'(u) = \ell_i(2t_i - u)$), preserves convexity, the list of admissible patterns, and strong robust non-imposition. A reflection maps a quota rule to a quota rule with σ_i replaced by $-\sigma_i$ and the same conventions. I write "up to a permutation" or "up to a reflection".

The proof has five steps. Locally, at most one coordinate changes label, and then the two others are strict and opposite (Lemma 11). Let K_i be the values where ℓ_i changes label; all three are non-empty (Lemma 13). For $t \in K_1 \times K_2 \times K_3$, the three axis-parallel lines through t miss U (Lemma 14), and U then meets exactly four orthants at t, arranged as a claw or a path (Lemma 15). A claw becomes a path when one threshold is moved (Lemma 16). A path gives a valid quota rule (Lemma 17).

Non-empty interior: the lemmas

Lemma 11 (Small box)**.** *Let $Q = Q_1 \times Q_2 \times Q_3 \subset U$ be a product of intervals and $\Lambda_i = \ell_i(Q_i)$. If Λ_1 has at least two elements, then $\Lambda_2 = \{\mu\}$ and $\Lambda_3 = \{\mu'\}$ with $\{\mu, \mu'\} = \{\mathsf{F}, \mathsf{S}\}$. The same holds for the other coordinates.*

Proof. The rule is valid at every point of Q, so this is Lemma 2 applied to Q. □

Definition 4 (Change points)**.** K_i is the set of $u \in E_i$ such that ℓ_i is constant on no neighbourhood of u.

Lemma 12. *If $J \subset E_i$ is an interval that does not meet K_i, then ℓ_i is constant on J.*

Proof. For each label λ, the set $\{u \in J : \ell_i(u) = \lambda\}$ is open in J, by local constancy. These three open sets partition J, which is connected, so one of them is J. □

Lemma 13 (The change points exist)**.** *K_1, K_2 and K_3 are non-empty.*

Proof. By strong robust non-imposition, $\{\ell_1 = \mathsf{F}\}$ and $\{\ell_1 = \mathsf{S}\}$ contain non-degenerate intervals of D_1, and a non-degenerate interval of D_1 meets its interior E_1. So ℓ_1 takes two values on the interval E_1, and by Lemma 12 E_1 meets K_1. The same holds for K_2 and K_3. □

Lemma 14 (Fibres above a change point)**.** *Let $u \in K_1$. Then $P_2^1(u)$ does not meet K_2, $P_3^1(u)$ does not meet K_3, ℓ_2 is constant on $P_2^1(u)$, ℓ_3 is constant on $P_3^1(u)$, and the two constants are F and S in some order. The same holds after permuting the coordinates. Consequently, if $t \in K_1 \times K_2 \times K_3$, the three lines through t parallel to the axes do not meet U.*

Proof. Let $v \in U$ with $v_1 = u$, and let $Q \subset U$ be a small cube centred at v. Its first factor is a neighbourhood of u, on which ℓ_1 is not constant. By Lemma 11, ℓ_2 is constant on the second factor and ℓ_3 on the third, with opposite strict values. So $v_2 \notin K_2$, $v_3 \notin K_3$, and $\{\ell_2(v_2), \ell_3(v_3)\} = \{\mathsf{F}, \mathsf{S}\}$. Every value of $P_2^1(u)$ is such a v_2, so $P_2^1(u) \cap K_2 = \emptyset$, and ℓ_2 is constant on the interval $P_2^1(u)$ by Lemma 12. The same holds for ℓ_3. The two constants are opposite, as one sees at any point of the fibre.

For the consequence, take the line $\{v_1 = t_1, v_2 = t_2\}$. A point of U on it would have $v_2 = t_2 \in K_2 \cap P_2^1(t_1)$, which has just been excluded. The two other lines are treated alike. □

Fix $t \in E_1 \times E_2 \times E_3$. For a sign vector $\varepsilon \in \{+,-\}^3$, let $O_\varepsilon = \{v : \varepsilon_i(v_i - t_i) > 0 \text{ for } i = 1,2,3\}$ be the open orthant at t, and let $\varepsilon^{(i)}$ be ε with coordinate i flipped. Let $M = \{\varepsilon : U \cap O_\varepsilon \neq \emptyset\}$. In what follows $\{i,j,k\} = \{1,2,3\}$.

Lemma 15 (Claw or path)**.** *Let U be open and convex, and $t \in E_1 \times E_2 \times E_3$ such that the three lines through t parallel to the axes miss U. Then:*

(i) for each i, the fibre $U \cap \{v_i = t_i\}$ lies in a single open quadrant: there are signs (s_j^i, s_k^i) with $s_j^i(v_j - t_j) > 0$ and $s_k^i(v_k - t_k) > 0$ on this fibre;

(ii) if ε and $\varepsilon^{(i)}$ are both in M, then $(\varepsilon_j, \varepsilon_k) = (s_j^i, s_k^i)$; and the two sign vectors with $(\varepsilon_j, \varepsilon_k) = (s_j^i, s_k^i)$ are both in M;

(iii) M has exactly four elements, and it is either a claw *or a* path*:*

$$M = \{\varepsilon, \varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}\} \quad \text{or} \quad M = \{\varepsilon,\ \varepsilon^{(p)},\ (\varepsilon^{(p)})^{(q)},\ -\varepsilon\},\ \{p,q,r\} = \{1,2,3\};$$

(iv) for a path, M contains neither $\varepsilon^{(q)}$ nor its opposite $-\varepsilon^{(q)}$.

Proof. (i) The fibre is convex, and non-empty since $t_i \in E_i$. It misses the lines $\{v_j = t_j\}$ and $\{v_k = t_k\}$ of the plane $\{v_i = t_i\}$. A convex set that misses a line of the plane lies in one open half-plane of it; so the fibre lies in a single open quadrant.

(ii) Let $p \in U \cap O_\varepsilon$ and $p' \in U \cap O_{\varepsilon^{(i)}}$. The segment $[p,p']$ lies in U. Coordinates j and k keep a constant strict sign along it, since both endpoints have the same strict sign and an open half-space is convex. Coordinate i changes sign, so the segment crosses the fibre $U \cap \{v_i = t_i\}$ at a point with signs $(\varepsilon_j, \varepsilon_k)$. By (i), these are (s_j^i, s_k^i). Conversely, a point r of this fibre lies in the open set U, and a small ball around r lies in U and meets both orthants with signs $(\pm, s_j^i, s_k^i)$ (in the order i,j,k).

(iii) Let G be the graph on M in which two sign vectors are joined if they differ in exactly one coordinate (the graph of the cube, restricted to M). By (ii), G has exactly one edge in each direction i, hence three edges. G is connected. Take $p \in U \cap O_\varepsilon$ and $p' \in U \cap O_{\varepsilon'}$. The segment $[p,p'] \subset U$ crosses each plane $\{v_i = t_i\}$ at most once, and never two planes at once, since a point on two of these planes lies on one of the three lines, which miss U. So the orthants it traverses form a walk in G from ε to ε'. The graph of the cube is bipartite, so G has no triangle. A connected graph with three edges and no triangle has four vertices and is a tree: a claw or a path. In the claw, the three edges leave the centre ε in three different directions. In the path, the three consecutive edges have the three different directions p,q,r, which gives the stated form, ending at $-\varepsilon$.

(iv) The vertices of the path are ε, $\varepsilon^{(p)}$, $(\varepsilon^{(p)})^{(q)}$, $-\varepsilon$. The sign vector $\varepsilon^{(q)}$ differs from each of them, and so does its opposite. □

Geometry alone does not exclude the claw. A small ball centred at $t + \delta(1,1,1)$ with radius r, $\delta < r < \delta\sqrt{2}$, crosses the three coordinate planes through t (at distance δ) but

misses the three lines (at distance $\delta\sqrt{2}$); it meets O_{+++} and its three neighbours. The labels do not exclude it either, but they allow one threshold to be moved to another change point (Lemma 16).

Lemma 16 (From claw to path)**.** *If $t \in K_1 \times K_2 \times K_3$ and M is a claw, there is $t' \in K_1 \times K_2 \times K_3$ for which M is a path.*

Proof. Up to reflections, the centre of the claw is $(+,+,+)$. By Lemma 15(ii), the edge of direction i joins $(+,+,+)$ to $(+,+,+)^{(i)}$, so $(s^i_j, s^i_k) = (+,+)$: each fibre $U \cap \{v_i = t_i\}$ lies in $\{v_j > t_j,\ v_k > t_k\}$.

For $j \neq i$, let $X^j_i = P^j_i(t_j)$, the projection on coordinate i of the fibre of U at $v_j = t_j$; thus $X^j_i \subset (t_i, +\infty)$. By Lemma 14 applied to $t_j \in K_j$, ℓ_i equals a strict constant L^j_i on X^j_i, and $L^j_i \neq L^j_k$.

If $L^2_1 = L^3_1$, $L^1_2 = L^3_2$ and $L^1_3 = L^2_3$, these three strict values would be pairwise different, because $L^3_1 \neq L^3_2$, $L^2_1 \neq L^2_3$ and $L^2_2 \neq L^1_3$. That is impossible with the two values F, S. Up to a permutation of the coordinates, which keeps the claw centred at $(+,+,+)$, $L^2_1 \neq L^3_1$.

Then X^2_1 and X^3_1 are disjoint. Take $x \in X^2_1$ and $y \in X^3_1$. Since $\ell_1(x) \neq \ell_1(y)$, Lemma 12 shows that the closed interval with endpoints x and y, which lies in E_1, contains a point $t'_1 \in K_1$. By Lemma 14, $t'_1 \notin X^2_1 \cup X^3_1$. These sets are intervals and t'_1 lies between a point of each, so they lie on either side of t'_1: there is $\alpha \in \{+,-\}$ with $\alpha(x - t'_1) > 0$ on X^2_1 and $-\alpha(x - t'_1) > 0$ on X^3_1.

Let $t' = (t'_1, t_2, t_3) \in K_1 \times K_2 \times K_3$. By Lemma 14, the three lines through t' miss U, so Lemma 15 applies at t'. Relative to t', the fibre at $v_2 = t_2$ has v_1 on side α and $v_3 > t_3$, so the edge of direction 2 joins $(\alpha,+,+)$ and $(\alpha,-,+)$. The fibre at $v_3 = t_3$ has v_1 on side $-\alpha$ and $v_2 > t_2$, so the edge of direction 3 joins $(-\alpha,+,+)$ and $(-\alpha,+,-)$. These two edges share no vertex. In a claw any two edges share the centre. So M is a path at t'. □

Lemma 17 (From path to quota)**.** *Let $t \in E_1 \times E_2 \times E_3$ be such that the three lines through t miss U and M is a path. Then there is a quota rule with thresholds t that is valid on V.*

Proof. By Lemma 15(iv), M misses two opposite orthants $\pm\rho$. Up to reflecting the coordinates where $\rho_i = -$ (by $v_i \mapsto 2t_i - v_i$, which fixes t), $\rho = (+,+,+)$. So U meets neither $O_{+++} = \{v > t\}$ nor $O_{---} = \{v < t\}$, inequalities being strict in every coordinate.

The fibres lie in mixed quadrants. For each i, the two orthants $(\pm, s^i_j, s^i_k)$ are in M (Lemma 15(ii)), and M contains neither $(+,+,+)$ nor $(-,-,-)$. So $s^i_j \neq s^i_k$.

From U to V. Since V is convex with non-empty interior, for $u \in U$ and $w \in V$ the half-open segment $[u, w)$ lies in U (Rockafellar, 1970, Theorem 6.1); so every point of V is a limit of points of U. Two consequences follow. First, V does not meet the open sets $\{v > t\}$ and $\{v < t\}$. Second, let $w \in V$ with $w_i = t_i$, and take u in the fibre $U \cap \{v_i = t_i\}$. The segment $[u, w)$ stays in this fibre, so w is in its closure, hence in the *closed* quadrant with signs (s^i_j, s^i_k). Since these signs differ, w has a coordinate $m \in \{j,k\}$ with $w_m \geq t_m$ and another with $w_{m'} \leq t_{m'}$.

Choice of conventions. Let $C = V \cap \{v \geq t\}$, with weak inequalities. It is convex. There is an i with $C \subset \{v_i = t_i\}$. Otherwise, pick for each i a point $p^i \in C$ with $p^i_i > t_i$; their barycentre with equal weights lies in C and has all coordinates above t, so it lies in $V \cap \{v > t\} = \emptyset$. If $C = \emptyset$, any i will do. Take $\sigma = (+,+,+)$, the convention "$>$" on coordinate i, and "$\geq$" on the two others j, k.

Verification. A point of V in $F_1 \times F_2 \times F_3$ is in C with $v_i > t_i$; there is none. A point $w \in V$ in $S_1 \times S_2 \times S_3$ has $w_i \leq t_i$, $w_j < t_j$, $w_k < t_k$. If $w_i < t_i$, it lies in $\{v < t\}$, which is excluded. If $w_i = t_i$, it has a coordinate $m \in \{j,k\}$ with $w_m \geq t_m$, which is excluded. So the quota rule is valid, and its thresholds lie in $E_i = \operatorname{int} D_i$. Back in the original coordinates, the reflections change only the orientations σ_i. □

Proof of Theorem 5(b). By Lemma 13, choose $t \in K_1 \times K_2 \times K_3$. By Lemma 14, the three lines through t miss U. By Lemma 15, M is a claw or a path. If it is a claw, Lemma 16 gives another point of $K_1 \times K_2 \times K_3$ at which it is a path. Lemma 17 then gives a quota rule valid on V, with thresholds in $K_1 \times K_2 \times K_3$. □

The proof uses neither validity on the boundary of V nor the labels at the endpoints of the D_i, and it does not show that a valid rule *is* a quota rule: on a range of v_1 where the two other coordinates are already opposite, ℓ_1 may be anything. It takes one change point per coordinate, reads the position of U around it, and derives a quota rule from it. No step uses the number of change points, which may be infinite.

Remark 2 (Convexity is needed). Let $\phi(u) = 0.4 + 0.2u$, applied coordinatewise, and let $V = \phi(R_1 \cup R_2 \cup R_3)$ with

$$R_1 = [0, 0.3] \times [0, 0.4] \times [0, 1], \quad R_2 = [0.4, 0.6] \times [0.6, 1] \times [0, 1], \quad R_3 = [0.7, 1] \times [0, 0.4] \times [0, 1].$$

Since $V \subset [0.4, 0.6]^3$, every point of V is the share vector of some distribution of citizens over strict rankings, and offering the triple exactly at the societies whose shares lie in V is a competition mechanism. The rule $\ell_1 = \mathsf{F}$ if and only if $v_1 \in [0.48, 0.52]$, $\ell_2 = \mathsf{F}$ if and only if $v_2 \leq 0.48$, $\ell_3 = \mathsf{F}$ if and only if $v_3 \geq 0.5$, and S otherwise, is valid: the first two labels are (S, F) on $\phi(R_1)$ and $\phi(R_3)$, and (F, S) on $\phi(R_2)$. It satisfies strong robust non-imposition. No quota rule is valid. Above every point of each box, v_3 takes every value in $[0.4, 0.6]$, so v_3 takes both labels of any quota, and a valid quota must give opposite labels to the first two coordinates on each box. Hence ℓ_1 takes the same value on $\phi([0, 0.3])$ and on $\phi([0.7, 1])$, the opposite of ℓ_2 on $\phi([0, 0.4])$. But a quota on coordinate 1 has one threshold in $(0.4, 0.6)$, so it labels 0.4 and 0.6 differently.

C.3 Four parties

With four parties $0, 1, 2, 3$, there is one coordinate per pair, ordered $01, 02, 03, 12, 13, 23$; the coordinate s_p of pair $p = \{i, j\}$, $i < j$, is the share of citizens who prefer i to j. Let $V \subset \mathbb{R}^6$ be convex and $D_p = \pi_p(V)$. A rule labels each value of each coordinate; it is valid at v if the six labels form a weak order. Transitivity involves three parties at a time, so validity is read triple by triple, on the cyclic patterns of the four triples. A quota rule picks, for each pair, a threshold in $\operatorname{int} D_p$, an orientation and a tie convention. Strong robust non-imposition asks, for each pair, that each strict ranking hold on a non-degenerate interval of D_p. To avoid fractions I write $S_p = 3s_p$.

Proposition 9 (A rule without a quota)**.** *Let V be the convex hull of the four points*

$$\tfrac{1}{3}(3, 0, 3, 0, 2, 1), \quad \tfrac{1}{3}(3, 1, 2, 2, 3, 2), \quad \tfrac{1}{3}(3, 2, 2, 2, 1, 2), \quad \tfrac{1}{3}(2, 0, 1, 1, 0, 2).$$

(i) For each triple, the projection of V on its three coordinates has non-empty interior in $\mathbb{R}^3$.

(ii) The rule

$$0 \succ 1 \iff S_{01} > \tfrac{8}{3}, \quad 0 \succ 2 \iff S_{02} \leq \tfrac{5}{3}, \quad 0 \succ 3 \iff S_{03} > \tfrac{8}{3},$$

$$1 \succ 2 \iff \tfrac{1}{3} \leq S_{12} \leq \tfrac{5}{3}, \quad 1 \succ 3 \iff S_{13} \leq \tfrac{1}{2}, \quad 2 \succ 3 \iff S_{23} < \tfrac{7}{6},$$

each pair being ranked strictly the other way otherwise, is valid at every point of V and satisfies strong robust non-imposition.

(iii) No quota rule, whatever its thresholds in $\operatorname{int} D_p$, orientations and conventions, is valid at every point of the relative interior of V.

Part (iii) is computer-assisted.

Proof. Write a point of V with barycentric weights $(a, b, c, d) \geq 0$, $a + b + c + d = 1$, on the four points in the order listed. Then

$$S_{01} = 3 - d, \;\; S_{02} = b + 2c, \;\; S_{03} = 2 + a - d, \;\; S_{12} = 2 - 2a - d, \;\; S_{13} = 2a + 3b + c, \;\; S_{23} = 2 - a.$$

The rule reads $0 \succ 1 \iff d < \frac{1}{3}$, $0 \succ 3 \iff a - d > \frac{2}{3}$, and $2 \succ 3 \iff a > \frac{5}{6}$. Two inequalities recur. First, $S_{02} = 2 - 2a - b - 2d$, so $S_{02} > \frac{5}{3}$ if and only if $2a + b + 2d < \frac{1}{3}$. Second, if $S_{13} \leq \frac{1}{2}$, then $c \leq \frac{1}{2} - 2a - 3b$, so $d = 1 - a - b - c \geq \frac{1}{2} + a + 2b$.

(i). Let A, B, C, D be the four points. On the three coordinates of the triples $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 2, 3\}$, $\{1, 2, 3\}$, the determinant of the projected vectors $3(B - A)$, $3(C - A)$, $3(D - A)$ equals 2, -2, -1, -2. None is zero.

(ii), validity. The rule ranks every pair strictly, so it is a tournament, and it suffices to exclude the eight cycles.

- $\{0, 1, 2\}$. The cycle $0 \succ 1 \succ 2 \succ 0$ needs $S_{02} > \frac{5}{3}$, that is $2a + b + 2d < \frac{1}{3}$, and $S_{12} \leq \frac{5}{3}$, that is $2a + d \geq \frac{1}{3}$; these are incompatible. The cycle $0 \succ 2 \succ 1 \succ 0$ needs $d \geq \frac{1}{3}$, hence $S_{12} \leq \frac{5}{3}$; so it needs $S_{12} < \frac{1}{3}$, that is $2a + d > \frac{5}{3}$, whence $a > \frac{2}{3}$ (as $d \leq 1 - a$) and then $d < \frac{1}{3}$: excluded.

- $\{0, 1, 3\}$. The cycle $0 \succ 1 \succ 3 \succ 0$ needs $S_{13} \leq \frac{1}{2}$, hence $d \geq \frac{1}{2}$, against $d < \frac{1}{3}$: excluded. The cycle $0 \succ 3 \succ 1 \succ 0$ needs $a > d + \frac{2}{3}$ and $d \geq \frac{1}{3}$, hence $a > 1$: excluded.

- $\{0, 2, 3\}$. The cycle $0 \succ 2 \succ 3 \succ 0$ needs $a > \frac{5}{6}$ and $a - d \leq \frac{2}{3}$, hence $d > \frac{1}{6}$ and $a + d > 1$: excluded. The cycle $0 \succ 3 \succ 2 \succ 0$ needs $a > \frac{2}{3}$ and $S_{02} > \frac{5}{3}$, which gives $a < \frac{1}{6}$: excluded.

- $\{1, 2, 3\}$. The cycle $1 \succ 2 \succ 3 \succ 1$ needs $a > \frac{5}{6}$, hence $S_{12} < \frac{1}{3}$ and $2 \succ 1$, against $1 \succ 2$: excluded. The cycle $1 \succ 3 \succ 2 \succ 1$ needs $S_{13} \leq \frac{1}{2}$, hence $a \leq \frac{1}{4}$ and $d \geq \frac{1}{2} + a + 2b$, so $S_{12} \leq \frac{3}{2} - 3a - 2b < \frac{5}{3}$; it therefore needs $S_{12} < \frac{1}{3}$, that is $2a + d > \frac{5}{3}$, while $2a + d \leq 1 + a \leq \frac{5}{4}$: excluded.

(ii), non-imposition. On V, $S_{01} \in [2, 3]$, $S_{02} \in [0, 2]$, $S_{03} \in [1, 3]$, $S_{12} \in [0, 2]$, $S_{13} \in [0, 3]$, $S_{23} \in [1, 2]$. Each strict ranking of each pair holds on a non-degenerate interval: for pair 12, $1 \succ 2$ on $[\frac{1}{3}, \frac{5}{3}]$ and $2 \succ 1$ on $[0, \frac{1}{3})$ and on $(\frac{5}{3}, 2]$; the other pairs have a single threshold interior to their range.

(iii). A quota rule is valid on the relative interior of V only if, for each triple T and each of its two cyclic orientations, no point of $\operatorname{relint} V$ lies in the *open* cyclic orthant where the three cyclic coordinates of T are strictly beyond their thresholds: such a point gives a cycle whatever the tie conventions. Fix an orientation $\sigma \in \{\pm 1\}^6$ of the six pairs; reversing all orientations exchanges the two cyclic orthants of every triple, so 32 classes suffice. For a triple T and one of its cyclic orthants $t_T + C$, with C an open orthant cone in the cyclic coordinates of T, the thresholds t_T for which $\operatorname{relint} V$ meets the orthant form $\pi_T(\operatorname{relint} V) - C = \pi_T(V) - C$ (Rockafellar, 1970, Theorem 6.6), the interior of the polyhedron $Q_T = \pi_T(V) - \operatorname{cl} C$. A threshold vector t avoids this set if and only if it satisfies some valid inequality $h \cdot t_T \geq \beta$ of Q_T, and a finite list of such inequalities containing all facets is obtained from the planes through three generators of Q_T. A quota rule valid on $\operatorname{relint} V$ with orientation σ would thus give a vector t, with $t_p \in \operatorname{int} D_p$, that satisfies for each of the eight pairs (triple, orthant) at least one inequality of the corresponding list. A depth-first search over these eight disjunctions closes every branch by a Motzkin certificate: non-negative rational multipliers that combine the inequalities of the branch into $0 \leq$ a negative number, or $0 < 0$. Checking a certificate is arithmetic. The certificates are found by floating-point linear programming and then rebuilt and checked in exact rational arithmetic; for this V, all 32 classes close, with 416 certificates.

An independent exact method, Fourier–Motzkin elimination over the rationals with strict inequalities tracked, reaches the same conclusion; in 28 of the 32 classes it reduces to an argument by hand, some vertex of V attaining on some triple the cyclic endpoint of each D_p. □

The rule has a *band* on pair 12: $1 \succ 2$ only for middle shares, $\frac{1}{3} \le S_{12} \le \frac{5}{3}$. All other pairs have a single threshold. The three ranges of pair 12 are forced, and by different triples. The low range is forced by $\{1,2,3\}$: the cycle $1 \succ 2 \succ 3 \succ 1$ is excluded only because $2 \succ 1$ when $S_{12} < \frac{1}{3}$. The high range is forced by $\{0,1,2\}$: the cycle $0 \succ 1 \succ 2 \succ 0$ is excluded only because $2 \succ 1$ when $S_{12} > \frac{5}{3}$. The band itself is used by both triples, in excluding $0 \succ 2 \succ 1 \succ 0$ and $1 \succ 3 \succ 2 \succ 1$. A quota rule makes no band, and the computation shows that no rearrangement of the other pairs removes it.

The set V itself is not produced by citizens: on its relative interior $s_{01} + s_{12} + s_{20} > 2$, whereas every citizen agrees with at most two duels of a cycle. The next lemma and Proposition 10 move it to a set that citizens do produce.

Shares can be moved towards one half without changing anything that matters to rules. Let $\varphi_\lambda(v)_p = \frac{1}{2} + \lambda(v_p - \frac{1}{2})$, $0 < \lambda \le 1$, applied to every coordinate of $v \in \mathbb{R}^{\binom{k}{2}}$.

Lemma 18 (Contraction)**.** *Let V be convex and ℓ a rule on V. The rule $\ell \circ \varphi_\lambda^{-1}$ on $\varphi_\lambda(V)$ has at $\varphi_\lambda(v)$ the pattern of ℓ at v. Validity, strong robust non-imposition, the hypothesis $D_p = \pi_p(V)$ and the existence of a valid quota rule hold on $\varphi_\lambda(V)$ if and only if they hold on V. If $\lambda \le \frac{1}{3}$, every point of $\varphi_\lambda([0,1]^{\binom{k}{2}})$ satisfies the triangle inequalities $1 \le s_{ij} + s_{jl} + s_{li} \le 2$.*

Proof. The map acts coordinate by coordinate and is affine and increasing. Labels travel with the coordinates, so patterns are unchanged point by point. Intervals go to intervals and interiors to interiors. A quota rule with threshold t_p, orientation and convention on V becomes the quota rule with threshold $\varphi_\lambda(t_p)$, the same orientation and the same convention, since $\varphi_\lambda(u) - \varphi_\lambda(t) = \lambda(u - t)$. The map fixes $\frac{1}{2}$ and commutes with $u \mapsto 1 - u$, so the cyclic coordinates $s_{li} = 1 - s_{il}$ are moved by the same map. For $\lambda \le \frac{1}{3}$, every coordinate lies in $[\frac{1}{3}, \frac{2}{3}]$, and every sum of three lies in $[1, 2]$. □

For four parties, the triangle inequalities and $0 \le s_p \le 1$ describe exactly the share vectors of distributions over the 24 strict rankings, the linear ordering polytope (Grötschel et al., 1985). In the paper's model citizens sit in the plane and the offer is a set of platforms. A fixed offer of four platforms makes only some rankings available, at most eighteen, so membership in this polytope is not enough, and producibility has to be checked for a given offer.

Proposition 10 (A producible set)**.** *Let $V_1 = \varphi_{1/4}(V)$, with V the set of Proposition 9.*

(i) The image of the rule of Proposition 9 is valid at every point of V_1 and satisfies strong robust non-imposition; no quota rule, with any thresholds, orientations and conventions, and no monotone rule with strong robust non-imposition, is valid at every point of $\operatorname{relint} V_1$.

(ii) Let the four platforms be $(0,0)$, $(1, \frac{1}{10})$, $(\frac{7}{20}, \frac{9}{10})$, $(\frac{13}{10}, \frac{3}{4})$. Every point of V_1 is the share vector of a society of finitely many groups of citizens in the plane with Euclidean preferences over these platforms.

Proof. (i) The rule and the absence of a quota rule follow from Proposition 9 and Lemma 18. For monotone rules: such a rule ranks each pair by a single threshold, interior to D_p, and an orientation. By the proof of Proposition 9(iii), for these thresholds and orientations some open cyclic orthant of a triple meets $\operatorname{relint} V_1$. There the three duels of the triple are strictly beyond their thresholds, so the monotone rule cycles.

(ii) Each vertex of V_1 has its coordinates in $[\frac{3}{8}, \frac{5}{8}]$. For each of eighteen rankings of the four platforms, an explicit rational position in the plane realizes it, with strict inequalities between the four squared distances checked in exact arithmetic, and each vertex is a convex combination, with rational weights checked exactly, of the share vectors of these eighteen rankings. The inequalities are strict, so a small disc around each position gives its citizens the same ranking. Citizens uniform on these discs, with these weights as masses, charge no line and produce the vertex, and mixtures of such societies produce the points of V_1. As in Remark 2, offering the four platforms exactly at the societies whose shares lie in V_1 is a competition mechanism. The certificates and scripts are available from the author. □

A second set, of full dimension in $\mathbb{R}^6$, carries a rule with a band and no quota rule, and grids of thresholds fail to detect the absence of a quota (Supplementary Appendix S5).

C.4 Families of strict rankings

The classical restricted domains are families of rankings. Let F be a set of strict rankings of k parties, x^o the share vector of the society unanimous on the ranking o, and $V = \mathrm{conv}\{x^o : o \in F\}$: the shares of the societies whose rankings all lie in F. For a triple T, let $F|_T$ be the set of restrictions of the rankings of F to T, and $\pi_T(V)$ the projection on the three cyclic coordinates of T. The six rankings of T are the corners of $\{0,1\}^3$ other than 000 and 111, and they form a hexagon in the graph of the cube: 110, 100, 101, 001, 011, 010 for ijl, ilj, lij, lji, jli, jil, in the coordinates (s_{ij}, s_{jl}, s_{li}). With k parties, a quota rule picks a threshold, an orientation and a tie convention for each pair, and it is valid if it is valid on every triple.

Proposition 11 (Families of rankings)**.** *Let ℓ be a rule valid on* $\mathrm{relint}\, V$ *with strong robust non-imposition, and T a triple with* $\dim \pi_T(V) = 3$*. Then:*

(i) $F|_T$ has exactly four rankings, consecutive on the hexagon: some party of T is never first in $F|_T$, or never last;

(ii) for every t in the product of the change points of the three pairs of T, the quota rule on T with thresholds t and the natural orientation, which ranks i above j when s_{ij} is above its threshold, is valid on $\mathrm{int}\, \pi_T(V)$*, with any conventions.*

If $\dim \pi_T(V) = 3$ *for every triple, then for every choice of one change point per pair, the quota rule with these thresholds and the natural orientation is valid on* $\mathrm{relint}\, V$*.*

Proof. Fix T and write $U = \mathrm{int}\, \pi_T(V)$, which equals $\pi_T(\mathrm{relint}\, V)$ (Rockafellar, 1970, Theorem 6.6). The restriction of ℓ to the three pairs of T is valid on U. The vertices of V are 0/1 vectors, and strong robust non-imposition forces each pair to take both values, so $D_p = [0,1]$; it also makes ℓ_p take both strict values on $(0,1)$, so by Lemma 12 the set K_p of change points in $(0,1)$ is non-empty. Take $t \in K_{ij} \times K_{jl} \times K_{li}$, with the change points of the cyclic coordinate $s_{li} = 1 - s_{il}$ read accordingly. By Lemma 14, which uses only validity on U, the three lines through t parallel to the axes miss U, and by Lemma 15 the set M of open orthants at t met by U has four elements and is a claw or a path.

Each point $x^{o|_T}$, $o \in F$, is a corner of the cube and lies in the open orthant O_δ, where $\delta_m = +$ if $x_m^{o|_T} = 1$ and $\delta_m = -$ otherwise, since $t \in (0,1)^3$. This orthant is a neighbourhood of a point of the closure of U, so it meets U, and $\delta \in M$. Distinct rankings give distinct δ, and a three-dimensional polytope has at least four vertices. So $F|_T$ has exactly four rankings, and M is the set of their δ. A claw $\{\varepsilon, \varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}\}$ always contains 000 or 111: if ε has one $+$, flipping it gives 000; if it has two, flipping the $-$ gives 111; if it has none or three, it is 000 or 111. These are not rankings, so M is a path. The six rankings induce a hexagon in the graph of the cube, so M is four consecutive corners of the hexagon, and four consecutive rankings are exactly those in which one party is never first or never last. This is (i).

For (ii): M contains neither 000 nor 111, so U meets neither $\{v < t\}$ nor $\{v > t\}$. Points of U on a threshold plane are handled as in the first step of Lemma 17: by Lemma 15(i)–(ii), the fibre $U \cap \{v_m = t_m\}$ lies in an open quadrant whose two signs differ, so the two other coordinates of such a point are strictly on opposite sides of their thresholds, and its pattern has an F and an S whatever the label of coordinate m. No point of U lies on two threshold planes. So the quota rule is valid on U whatever the conventions. In cyclic coordinates its orientation is $(+,+,+)$, which is the natural orientation on each pair, since s_{li} above t_{li} means s_{il} below $1 - t_{li}$.

For the last claim, a point of $\operatorname{relint} V$ projects into $\operatorname{int} \pi_T(V)$ for every T, and the quota rule is natural on every triple: there is no orientation to reconcile across triples. □

The boundary of V and the flat triples are settled on Condorcet domains by one observation, which also gives the tie-breaking of Section 6. Given non-negative masses of total 1 on the rankings of a triple, s_{xy} is the total mass of the rankings that put x above y.

Lemma 19 (Incompatible relations)**.** *Let S be a set of rankings of a triple without a Latin square. In every cycle $x \to y \to z \to x$ on the triple, two of the three relations hold together in no ranking of S. Hence, for masses carried by S:*

(i) the shares of these two relations add up to at most 1, and s_{xy}, s_{yz}, s_{zx} are not all above $\frac{1}{2}$;

(ii) if $\tau \in S$, majority with exact ties broken by τ has no cycle.

Proof. A ranking satisfies at least one and at most two of the relations $x \succ y$, $y \succ z$, $z \succ x$, and those that satisfy two are xyz, yzx, zxy. These three form a Latin square, so one of them is missing from S; say xyz. Then no ranking of S has both $x \succ y$ and $y \succ z$, so $s_{xy} + s_{yz} \leq 1$, which gives (i). For (ii), suppose majority with ties broken by τ gives the cycle $x \succ y \succ z \succ x$. Then $s_{xy}, s_{yz} \geq \frac{1}{2}$, so both are exactly $\frac{1}{2}$, and both ties are decided by τ. So τ has $x \succ y$ and $y \succ z$, that is $\tau = xyz \notin S$, a contradiction. □

Proposition 12 (Condorcet domains)**.** *Let F be a family of strict rankings of k parties in which no pair is unanimous and no triple carries a Latin square.*

(i) For any $o \in F$, majority with exact ties broken by o is valid on the closed set V. It is the natural quota rule with all thresholds at $\frac{1}{2}$ and conventions read on o, and it satisfies strong robust non-imposition.

(ii) Suppose that every $F|_T$ consists of four consecutive rankings. Let p be positive weights on F of total 1, p_{xy} the weight of the rankings that put x above y, and $\delta \in (0, 1]$. The natural quota rule with thresholds $t_{xy} = \frac{1}{2} + \delta(\frac{1}{2} - p_{xy})$ is valid on the closed set V whatever the conventions: the closed cyclic orthants of every triple miss $\pi_T(V)$.

(iii) Suppose that some $F|_T$ consists of two opposite edges of the hexagon. Then for every natural quota rule, a closed cyclic orthant of T meets $\pi_T(V)$: the conventions cannot be dispensed with.

Conversely, if a triple of a family F carries a Latin square, majority has a strict cycle on V, however ties are treated. So on a family without a unanimous pair, majority with a suitable tie-break is valid on V if and only if the family is a Condorcet domain.

Proof. (i) On each triple T, the shares are carried by $F|_T$, which has no Latin square and contains the restriction of o; apply Lemma 19(ii). Since no pair is unanimous, $D_p = [0, 1]$ for each pair, and $\frac{1}{2}$ is interior to it.

(ii) Note that $t_{yx} = 1 - t_{xy}$, and that $t_{xy} \in (0,1)$ because $p_{xy} \in (0,1)$. A closed cyclic orthant of T is the set of share vectors with $s_{xy} \geq t_{xy}$, $s_{yz} \geq t_{yz}$, $s_{zx} \geq t_{zx}$ for one of the two cycles on T: being below all three thresholds of a cycle is being above all three of the reverse cycle. By Lemma 19, two relations of the cycle are incompatible on $F|_T$; say $xyz \notin F|_T$, so that $s_{xy} + s_{yz} \leq 1$ on V and $p_{xy} + p_{yz} \leq 1$. Four consecutive rankings leave out two adjacent corners of the hexagon, and xyz and zyx are opposite, so $zyx \in F|_T$. The rankings that restrict to zyx have neither relation and positive weight, so $p_{xy}+p_{yz} < 1$. Hence $t_{xy} + t_{yz} = 1 + \delta(1 - p_{xy} - p_{yz}) > 1 \geq s_{xy} + s_{yz}$, and the orthant misses $\pi_T(V)$.

(iii) Let y be the party never in the middle. The rankings are yxz, yzx, xzy, zxy, so in the coordinates (s_{xy}, s_{yz}, s_{zx}) the set $\pi_T(V)$ is $\{(u, 1-u, w) : u, w \in [0,1]\}$. If $t_{xy} + t_{yz} \leq 1$, take $u \in [t_{xy}, 1 - t_{yz}]$ and $w = 1$: the point is above all three thresholds. Otherwise take $u \in [1 - t_{yz}, t_{xy}]$ and $w = 0$: it is below all three.

For the converse, put mass $\frac{1}{3}$ on three rankings of F that restrict to ijl, jli, lij. Then $s_{ij} = s_{jl} = s_{li} = \frac{2}{3}$. □

The rule of (ii) is majority among phantom voters as well as real ones: x goes above y when $s_{xy} > t_{xy}$, that is, when x has a majority in the society mixed with a fixed society of weight δ whose rankings are distributed by p. The device is known. It goes back to Moulin (1980, 1984) on single-peaked domains, where anonymous and monotone Arrovian rules are of this form. On a closed Condorcet domain with n voters, Puppe and Slinko (2026, Theorem 10.1.5) add $n-1$ phantom voters with rankings in the domain, which gives the threshold $n - n_{xy}$ for x above y, where n_{xy} counts the phantoms who put x above y; the case $\delta = 1$ is its limit as n grows, with phantoms distributed by p. The total is odd there, so no tie arises. What the proposition adds is the continuum, where exact ties do arise, and what then decides whether conventions are needed: (ii) dispenses with them under four consecutive rankings on every triple and positive weights, while by (iii) a flat triple of two opposite edges brings them back. Closure serves there to keep the social ranking in the domain, which is not asked here. The thresholds of (ii) respect the inclusions of Section 5: if every ranking of F with x above y has u above v, then $p_{xy} \leq p_{uv}$ and $t_{xy} \geq t_{uv}$. For instance, for the eight rankings single-peaked on the axis $1,2,3,4$, with equal weights and $\delta = 1$, the thresholds of $1 \succ 2$, $1 \succ 3$, $1 \succ 4$, $2 \succ 3$, $2 \succ 4$, $3 \succ 4$ are $\frac{7}{8}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. The cycle $1 \succ 2 \succ 3 \succ 1$ would need $s_{12} \geq \frac{7}{8}$ and $s_{31} \geq \frac{1}{4}$, which add up to more than 1, whereas no ranking has both $1 \succ 2$ and $3 \succ 1$, since 2 is never last in $\{1,2,3\}$.

Remark 3 (Families of rankings). (i) A set of rankings of a triple without a Latin square (Ward's condition, Ward, 1965) has at most four elements, and the sets of four are the six paths of consecutive rankings, of dimension 3, and the three pairs of opposite edges of the hexagon, which are flat and in which some party is never in the middle. For a triple of dimension 3, condition (i) of the proposition is thus exactly Ward's condition, and a three-dimensional $F|_T$ with five or six rankings, or with four that are not consecutive, carries no rule valid on the relative interior with strong robust non-imposition: the impossibility, read on families of rankings, comes from counting orthants rather than from free arbitration. On four consecutive rankings, the exact ties that make weak majority intransitive (masses $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ on ijl, jli, lji give $i \sim j$, $j \succ l$, $i \sim l$) arise only on the boundary, where one of the four rankings has no mass. Let m be never last and u, v the two others. Every ranking with u above m has u above v, and muv has u above v but not above m; so when the four rankings have positive mass, $s_{um} < s_{uv}$, and likewise $s_{vm} < s_{vu}$. A weak majority cycle needs $s_{um}, s_{vu} \geq \frac{1}{2}$ or $s_{vm}, s_{uv} \geq \frac{1}{2}$, which these inequalities exclude. If m is never first, the same argument runs with $s_{mv} < s_{uv}$ and $s_{mu} < s_{vu}$. Hence, when every restriction consists of four consecutive rankings, majority is valid on $\operatorname{relint} V$, with exact ties left as indifference or broken by any convention.

(ii) When every restriction consists of four consecutive rankings, a natural quota rule is valid on the closed set V whatever the conventions, for any number of parties (Proposition 12(ii)). With a flat triple whose rankings are two opposite edges of the hexagon, there is none, and the conventions of Proposition 12(i) are what makes the rule valid.

(iii) Proposition 11 says nothing about triples of dimension two or less, where the natural quota at the rule's own change points need not be valid, nor about the boundary of V. On a Condorcet domain without a unanimous pair neither matters for existence: a quota rule exists on the closed set, for any number of parties (Proposition 12(i)). With a unanimous pair no rule satisfies strong robust non-imposition. What remains are the families with a triple that is a flat Latin square. With four parties they are settled. Of the 455 families that could carry a rule, up to relabelling and to the reversal of all rankings, the social one included, 414 are Condorcet domains and 41 are not. Of these 41, 39 carry a natural quota rule valid on V; these rules were found by computation and checked in exact arithmetic. The two others, $\{1234, 1324, 2413, 3412\}$ and $\{1234, 1342, 2341, 4213\}$, carry no rule valid on $\operatorname{relint} V$ with strong robust non-imposition, whatever its breakpoints (Proposition S16 in the Supplementary Appendix). So with four parties a family carries a rule if and only if it carries a quota rule, and then a natural one. For five parties or more, whether a family with a flat Latin square can carry a rule without a quota is open.

(iv) Both hypotheses of the proposition are needed for its conclusion. Dimension: if $F|_T$ is a Latin square, the quota 2/3 with conventions $\geq, \geq, >$ on the cyclic coordinates is valid on the closed set, natural, and satisfies strong robust non-imposition and Pareto, while majority cycles; only neutrality fails. Strong robust non-imposition: let x, y, z be an order of precedence, and let a go above b, for a before b, unless $s_{ab} = 0$. This is unanimity with an order of precedence, and it is valid on the closed set as soon as $F|_T$ does not contain both yzx and zxy (Dasgupta and Maskin, 2003, Lemma 3), so with up to five rankings. It satisfies Wilson's axiom when F contains, for each pair, a ranking each way, but it ranks b above a at the single share $s_{ab} = 0$.

D Proofs for Section 6

Every object in this appendix depends on a profile x only through its distribution, which I write μ instead of μ_x. By the standing assumption, μ charges no line. Write $K(\mu)$ for the closure of $\operatorname{conv}(\operatorname{supp}\mu)$. A convex set of the plane with empty interior lies in a line, so $\operatorname{conv}(\operatorname{supp}\mu)$ has non-empty interior, hence the same interior as its closure: $\operatorname{int} K(\mu)$ is the set that appears in the definition of a separated offer. For $a \neq b$, H_{ab} is the open half-plane of points closer to a than to b, $s_{ab}(\mu) = \mu(H_{ab})$, and $s_{ab} + s_{ba} = 1$ since the bisector is null. For a non-aligned triple T, ω_T is its circumcentre.

By Lemma 1, an anonymous and independent rule is a labelling: it ranks a and b according to a label $\ell_{ab}(s_{ab}(\mu)) \in \{\mathsf{F}, \mathsf{S}, \mathsf{I}\}$, with $\ell_{ba}(1-w)$ opposite to $\ell_{ab}(w)$, and the label may depend on the positions of a and b. Conversely, every labelling defines an anonymous and independent rule, since the shares depend on the profile only through μ and do not change when $\{i : a \succ_i b\}$ changes by a null set. I therefore work with labellings throughout. A labelling is *valid* on a domain if it gives a weak order on every offer of the domain. A labelling is a *strict total order* on an offer if it gives a linear order on it.

D.1 The geometry of a separated triple

Lemma 20 (Half-plane)**.** *Let T be a non-aligned triple. T is separated for μ, that is $\omega_T \notin \operatorname{int} K(\mu)$, if and only if there is an open half-plane P, bounded by a line through ω_T, with $\mu(P) = 1$.*

Proof. Let $K = K(\mu)$. If $\omega_T \notin \operatorname{int} K$, some line L through ω_T leaves K in one of its closed sides: the parallel through ω_T of a line that separates ω_T from K if $\omega_T \notin K$, a supporting line of K at ω_T otherwise. Let P be the open side of L on which K lies. The support lies in $P \cup L$ and $\mu(L) = 0$, so $\mu(P) = 1$. Conversely, if $\mu(P) = 1$, the support, the smallest closed set of full mass, lies in $\bar{P}$. Hence $K \subset \bar{P}$ and $\operatorname{int} K \subset \operatorname{int} \bar{P} = P$, which does not contain ω_T. □

Fix a triple T. A citizen who is not on a bisector ranks T strictly. The six strict orders are written as words: *abc* ranks a first and c last. Let $\pi_\mu(o)$ be the mass of citizens whose ranking of T is o. Bisectors are null, so $\sum_o \pi_\mu(o) = 1$ and $s_{xy}(\mu)$ is the total mass of the orders that put x before y. The orders *carried* by μ form the set $\Omega_T(\mu) = \{o : \pi_\mu(o) > 0\}$. For $X \in T$, $W_X^\uparrow$ is the set of the four orders in which X is not first, and $W_X^\downarrow$ the set of the four orders in which X is not last. These six sets are the *windows*. A window of type $W^\downarrow$ is a window in the sense of Section 6.

Lemma 21 (Windows)**.** *Let $T = \{a, b, c\}$.*

(i) If T is aligned with b in the middle, every citizen off the bisectors has an order in $W_b^\downarrow$.

(ii) If T is not aligned and P is an open half-plane bounded by a line through ω_T, the orders of the citizens of P off the bisectors all lie in one window $W(P)$. If P meets only three of the six sectors of Lemma 6(a), $W(P)$ can be chosen of type $W^\downarrow$.

Hence, if T is aligned or separated for μ, $\Omega_T(\mu)$ lies in a window. If moreover each member of T is ranked first by a positive mass of citizens, this window can be chosen of type $W^\downarrow$: some party is never ranked last.

Proof. (i) Parametrize the line of T by t, with $a < b < c$. For a citizen z with projection $t(z)$ on this line, $|z - x|^2 - |z - y|^2$ is an affine function of $t(z)$ that vanishes at the midpoint of $[x, y]$. If b were last for z, then $t(z) < (a + b)/2$, since z prefers a to b, and $t(z) > (b + c)/2$, since z prefers c to b. This is impossible, since $(a + b)/2 < (b + c)/2$.

(ii) Write $\omega = \omega_T$. By Lemma 6(a), the three bisectors cut the plane into six open sectors, bounded by six half-lines issued from ω, and in angular order the sectors run through the hexagon of rankings: consecutive sectors carry orders that differ by one exchange of adjacent places, and opposite sectors carry reversed orders. The exchanges alternate between the two first places and the two last places, since two successive exchanges of the same places would return to the starting order.

P is an open angle π at ω, and every line through ω other than its boundary has exactly one half-line in P. If the boundary of P contains no bisector, P contains three consecutive half-lines, meets the four sectors adjacent to them and misses the two others, which are consecutive and lie in the opposite closed half-plane. If the boundary is a bisector, P contains two half-lines, meets three consecutive sectors and misses the three others. The orders of the citizens of P off the bisectors are the orders of the sectors met. If P misses two sectors, their orders differ by one exchange. If it bears on the two first places, both orders end with the same party X: they are the two orders in which X is last, and the orders met lie in $W_X^\downarrow$. If it bears on the two last places, both orders start with the same X, and the orders met lie in $W_X^\uparrow$. If P misses three consecutive sectors, the two exchanges between them alternate, so one of the two consecutive pairs among them differs by an exchange of the two first places, and the orders met lie in $W_X^\downarrow$, where X is last in both orders of this pair.

For the last two sentences, if T is aligned, (i) gives the window $W_b^\downarrow$. If T is separated, Lemma 20 gives P with $\mu(P) = 1$, and $\Omega_T(\mu) \subset W(P)$ by (ii). If $W(P) = W_X^\uparrow$, then X is never ranked first, which is excluded when each party is ranked first by a positive mass; so $W(P)$ is then of type $W^\downarrow$. □

Lemma 22 (No strict cycle on a window)**.** *Let π be non-negative masses on the six orders of a triple, of total* 1 *and carried by a window, and let s_{xy} be the total mass of the orders that put x before y. There is no cycle $x \succ y \succ z \succ x$ with s_{xy}, s_{yz}, s_{zx} all strictly above $\frac{1}{2}$.*

Proof. A Latin square ranks each party first once and last once, so a window, in which some party is never first or never last, contains none. Apply Lemma 19(i). □

These are two of the three forms of the value restriction of Sen (1966); the proof above does not depend on the citation.

D.2 Majority without exact ties

Theorem 6 (No exact ties)**.** *Let the mechanism be separating, and suppose that $s_{ab}(\mu) \neq \frac{1}{2}$ for every $(x, O) \in \mathcal{D}$ and all $a \neq b$ in O. Then strict majority rule is a strict total order on every offer. It is anonymous, independent and non-null. It satisfies non-imposition if and only if, for every offered pair $\{a, b\}$, there are (x, O) and (x', O') in $\mathcal{D}$ that offer a and b with $s_{ab}(\mu_x) > \frac{1}{2} > s_{ab}(\mu_{x'})$.*

Proof. Without ties, majority is complete and asymmetric on O: it is a tournament. A tournament without a cycle of length three is a linear order. Indeed, if $x \succ y \succ z$ and not $x \succ z$, then $z \succ x$ by completeness, and x, y, z form a cycle. Let $T \subset O$ be a triple. It is aligned or, by the definition of a separating mechanism, separated for μ. By Lemma 21, $\Omega_T(\mu)$ lies in a window, and by Lemma 22 there is no cycle on T. Majority is a labelling, hence anonymous and independent. It is non-null because every offer has at least two platforms and majority is strict. It is strict, so "a is ranked weakly above b at some profile" means "$s_{ab} > \frac{1}{2}$ at some profile", which gives the condition for non-imposition. □

D.3 Exact ties

A *strict tie-break* on a triple T is a tournament τ on T. "Majority with ties broken by τ" ranks x above y if $s_{xy} > \frac{1}{2}$, or if $s_{xy} = \frac{1}{2}$ and $x \,\tau\, y$. It is a tournament on T, so it is transitive if and only if it has no cycle.

Lemma 23 (Tie-breaking in a window)**.** *Let W be a window of T and τ a tournament on T. Majority with ties broken by τ is transitive for all masses of total* 1 *carried by W if and only if τ is a linear order and belongs to W.*

Proof. If. A window contains no Latin square, as in the proof of Lemma 22; apply Lemma 19(ii).

Only if. Reversing every order maps masses carried by $W_X^{\uparrow}$ to masses carried by $W_X^{\downarrow}$, maps each share s_{xy} to $1 - s_{xy}$, so keeps the exact ties, and reverses both τ and the relation it produces; and $\tau \in W_X^{\uparrow}$ if and only if the reverse of τ lies in $W_X^{\downarrow}$. A cycle on $W_X^{\uparrow}$ for some $\tau \notin W_X^{\uparrow}$ thus gives a cycle on $W_X^{\downarrow}$ for the reverse of τ, and it suffices to treat $W = W_X^{\uparrow}$. Let $\tau \notin W_X^{\uparrow}$: τ is cyclic, or a linear order with X first. The two other parties can be named Y, Z so that $X \,\tau\, Y$ and $Y \,\tau\, Z$: this covers $\tau = XYZ$ and the cycle $X \to Y \to Z \to X$, and the exchange of the names covers $\tau = XZY$ and the cycle $X \to Z \to Y \to X$. For $0 < r < \frac{1}{2}$, take

$$\pi(ZXY) = \tfrac{1}{2}, \qquad \pi(YZX) = r, \qquad \pi(YXZ) = \tfrac{1}{2} - r,$$

three orders of $W_X^{\uparrow}$. Then $s_{XY} = \frac{1}{2}$, $s_{YZ} = \frac{1}{2}$ and $s_{ZX} = \frac{1}{2} + r$. The ties give $X \succ Y$ and $Y \succ Z$, and majority gives $Z \succ X$: a cycle. □

The next proposition says that no treatment of ties works for a fixed triangle. Let $T = \{a, b, c\}$ be non-aligned and let $\mathcal{D}_T$ be the set of pairs (x, T) such that T is separated for μ_x: the offer is always T, and the profiles are all those for which T is separated. Let $\mathcal{D}_T^+ \subset \mathcal{D}_T$ be the subset in which each cell of T has positive mass, that is, each party of T is ranked first by a positive mass. Both are mechanisms in the sense of Section 2, since separation depends on μ_x only, and both are separating. Lemma 1 therefore applies to them.

Proposition 13 (Fixed triangle)**.** *On $\mathcal{D}_T$, no rule satisfies anonymity, independence, non-imposition and non-nullness.*

Proof. Realization. I first show that for each window W of T and all masses π on W of total 1, some $(x, T) \in \mathcal{D}_T$ has $\pi_{\mu_x} = \pi$, and that $(x, T) \in \mathcal{D}_T^+$ when each party is ranked first by a positive mass under π. The two orders outside W differ by the exchange of the two first places (if $W = W_X^{\downarrow}$) or of the two last places (if $W = W_X^{\uparrow}$), so by Lemma 6(a) they are carried by two consecutive sectors. The union U of these two sectors and of the half-line between them is an open angle at ω_T bounded by two half-lines h_k, h_{k+2} of the angular order; the half-line opposite to h_k is h_{k+3}, so its opening is less than π. Among the lines through ω_T that leave $\bar{U}$ in one closed side, a set of directions of positive length, pick one, L, that is not a bisector, and let P be the open side of L that does not meet U. The boundary of P contains no bisector, so P meets four sectors (proof of Lemma 21); it misses the two sectors of U, so it meets exactly the four sectors that carry the orders of W. For each $o \in W$, place a small closed disc D_o in the open non-empty set $P \cap$ (sector of o), let $\mu = \sum_{o \in W} \pi(o) \operatorname{Unif}(D_o)$, and let x be a profile with distribution μ (every probability on $\mathbb{R}^2$ is the distribution of a measurable map on I). Then μ charges no line, $\mu(P) = 1$, so T is separated for μ by Lemma 20, and $\pi_\mu = \pi$.

Impossibility. Write share vectors as (s_{ab}, s_{bc}, s_{ca}), let V be the set of share vectors produced on $\mathcal{D}_T$, and let $h = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. In the window $W_b^{\downarrow}$, with masses p, q, r, t on abc, bac, bca, cba, the shares are $(p, 1 - t, r + t)$. For $u \in [0, \frac{1}{2}]$ and $v \in [\frac{1}{2}, 1]$, the masses $p = u$, $q = \frac{1}{2} - u$, $r = v - \frac{1}{2}$, $t = 1 - v$ are non-negative, sum to 1 and give $(u, v, \frac{1}{2})$. By the realization, the product $[0, \frac{1}{2}] \times [\frac{1}{2}, 1] \times \{\frac{1}{2}\}$ lies in V, and it contains h. Rotating the names $(a \to b \to c \to a)$, the windows $W_c^{\downarrow}$ and $W_a^{\downarrow}$ give the products $\{\frac{1}{2}\} \times [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ and $[\frac{1}{2}, 1] \times \{\frac{1}{2}\} \times [0, \frac{1}{2}]$ in V. The three products contain h. On each coordinate their factors cover $[0, 1]$, and each factor lies in the corresponding projection of V, so the projections D_{ab}, D_{bc}, D_{ca} all equal $[0, 1]$ and are covered. Since the offer is always T, non-imposition bears on these projections. By Lemma 1 an anonymous and independent rule is a labelling, and by Lemma 3 a valid non-imposed labelling is null on T, which is the whole offer. □

Two complements. First, with the profiles that have an exact tie removed, $\mathcal{D}_T$ is still separating, and Theorem 6 makes strict majority rule valid, non-null and non-imposed, since each pair is won strictly both ways: for $0 < \delta < \frac{1}{8}$, put mass $1 - 3\delta$ on one order of $W_b^{\downarrow} = \{abc, bac, bca, cba\}$ and δ on each of the three others. Every share is at most 3δ or at least $1 - 3\delta$, so none is $\frac{1}{2}$; the large mass on abc, bac, bca, cba gives $s_{ab} > \frac{1}{2}$, $s_{ab} < \frac{1}{2}$, $s_{bc} > \frac{1}{2}$, $s_{bc} < \frac{1}{2}$ respectively, and cba gives $s_{ca} > \frac{1}{2}$, abc gives $s_{ca} < \frac{1}{2}$. Each party is ranked first by a positive mass in these profiles, so the same holds on $\mathcal{D}_T^+$. Second, the proof of the proposition rests on the triple tie h, which a window produces only with mass $\frac{1}{2}$ on each of two reversed orders; their middle party is then ranked first by nobody. On $\mathcal{D}_T^+$, h is not produced and the argument does not apply. Majority rule is nevertheless not valid there, whatever labels the exact ties receive (Proposition S17 in the Supplementary Appendix).

D.4 The arbiter

Definition 5 (Arbiter)**.** An *arbiter* A is a point $z_0 \in \mathbb{R}^2$ or a unit direction d. Its order on $\mathbb{R}^2$ is defined as follows. For a point z_0: $x \succ_{z_0} y$ if $|x - z_0| < |y - z_0|$, or if equality holds and

$x_1 > y_1$, or if equality holds again and $x_2 > y_2$. For a direction d: $x \succ_d y$ if $\langle x, d\rangle > \langle y, d\rangle$, or if equality holds and $x \succ_0 y$, the order of the point 0. The rule M_A is strict majority with exact ties broken by $\succ_A$: a is ranked above b if $s_{ab} > \frac{1}{2}$, or if $s_{ab} = \frac{1}{2}$ and $a \succ_A b$.

Both orders are linear orders on $\mathbb{R}^2$: if $|x - z_0| = |y - z_0|$ and $x_1 = y_1$ with $x \neq y$, then $x_2 \neq y_2$. They are fixed: they depend neither on the pair nor on the offer. So M_A is a labelling, whose tie label for (a, b) depends only on the positions of a and b, and it is anonymous and independent. Both arbiters are the rules described before Theorem 3; the fixed convention there is the one of this definition. In words, the point arbiter is a citizen at z_0 with a fixed rule for her own indifferences, and the direction arbiter is a citizen "at infinity" in the direction d.

Lemma 24 (Realizing the arbiter)**.** *Let Q be a finite set of points and P an open half-plane with inward normal n. Suppose $z_0 \in P$ (point arbiter) or $\langle d, n\rangle > 0$ (direction arbiter). There is a point $z \in P$, on none of the bisectors of Q, that ranks Q by $\succ_A$.*

Proof. Point arbiter. Let $z_\eta = z_0 + \eta e_1 + \eta^2 e_2$. For $x \neq y$,

$$|z_\eta - x|^2 - |z_\eta - y|^2 = (|z_0 - x|^2 - |z_0 - y|^2) + 2\eta(y_1 - x_1) + 2\eta^2(y_2 - x_2).$$

For small $\eta > 0$, the sign is the sign of the first non-zero term, and it is negative exactly when $x \succ_{z_0} y$. Since Q is finite, one η works for all pairs. Then z_η is off the bisectors, and it lies in P for small η because P is open.

Direction arbiter. By the first part, applied to the point 0, there is y off the bisectors of Q that ranks Q by $\succ_0$. For $z = y + \lambda d$ and $x \neq x'$ in Q,

$$|z - x|^2 - |z - x'|^2 = |y - x|^2 - |y - x'|^2 - 2\lambda\langle d, x - x'\rangle.$$

For large λ, the sign is given by $\langle d, x\rangle$ against $\langle d, x'\rangle$ when they differ, and by $\succ_0$ when they are equal; it is never zero. Let p be a point on the boundary of P. Then $\langle z - p, n\rangle = \langle y - p, n\rangle + \lambda\langle d, n\rangle > 0$ for large λ, so $z \in P$. □

Theorem 7 (Arbiter)**.** *Let the mechanism be separating and let A be an arbiter. Assume*

(A) for every $(x, O) \in \mathcal{D}$ and every non-aligned triple $T \subset O$ with at least one exactly tied pair, there is an open half-plane P, bounded by a line through ω_T, with $\mu_x(P) = 1$, that contains the arbiter: $z_0 \in P$, or $\langle d, n\rangle > 0$ for the inward normal n of P.

Then M_A is a strict total order on every offer. It is anonymous, independent and non-null. It satisfies non-imposition if and only if, for every offered pair $\{a, b\}$, there are profiles in $\mathcal{D}$ that offer a and b at which $a \succ b$ and $b \succ a$ respectively; it suffices that $s_{ab} > \frac{1}{2}$ at one and $s_{ab} < \frac{1}{2}$ at the other.

Proof. M_A is a tournament on O, since exact ties are broken by a linear order. It suffices to exclude a cycle on each triple $T \subset O$. If T has no exact tie, its relation is strict majority, and the argument of Theorem 6 applies to T. Suppose T has an exact tie. If T is aligned with b in the middle, $\Omega_T(\mu) \subset W_b^{\downarrow}$ by Lemma 21(i). Lemma 24, applied with any open half-plane that contains the arbiter, gives a citizen off the bisectors whose order on T is $\succ_A$; by Lemma 21(i), this order lies in $W_b^{\downarrow}$. If T is not aligned, let P be given by (A). The orders carried are those of the sectors met by P, which lie in $W(P)$ by Lemma 21(ii). Lemma 24 gives $z \in P$, off the bisectors, that ranks T by $\succ_A$. The sector of z meets P, so the restriction of $\succ_A$ to T lies in $W(P)$. In both cases the restriction of $\succ_A$ to T is a linear order in a window that contains $\Omega_T(\mu)$, and Lemma 23 (the "if" part) excludes a cycle. M_A is a labelling, hence anonymous and independent; it is strict and every offer has two platforms, so it is non-null, and non-imposition reads as stated. □

Remark 4. Condition (A) serves only to put the arbiter's order on T in a window that contains $\Omega_T(\mu)$. By Lemma 19(ii) it would even suffice that $\Omega_T(\mu) \cup \{\succ_A |_T\}$ contains no Latin square; (A) is a convenient sufficient condition, not a necessary one. Proposition 13 shows what happens when no arbiter is on the right side for all societies: the windows $W_a^{\downarrow}$, $W_b^{\downarrow}$, $W_c^{\downarrow}$ are all produced, and no order lies in all three.

Corollary 2 (Interior arbiter)**.** *If $z_0 \in \operatorname{int} K(\mu_x)$ for every $(x, O) \in \mathcal{D}$, then (A) holds for every separating mechanism on $\mathcal{D}$, and Theorem 7 applies. This is the case if every μ_x has a finite first moment and barycentre z_0.*

Proof. Let T be separated for μ and P given by Lemma 20. The support lies in $\bar{P}$, so $\operatorname{int} K(\mu) \subset \operatorname{int} \bar{P} = P$, and $z_0 \in P$. For the second sentence, let g be the barycentre of μ and suppose $g \notin \operatorname{int} K(\mu)$. A separating or supporting line gives $u \neq 0$ with $\langle u, y - g\rangle \leq 0$ on $K(\mu)$. Integrating, $0 = \int \langle u, y - g\rangle \, d\mu(y)$, with an integrand that is non-positive μ-almost everywhere. So the integrand vanishes almost everywhere, and μ is carried by the line $\{\langle u, y - g\rangle = 0\}$, which is excluded. □

Proof of Theorem 3. (i) is Theorem 6. (ii) is Theorem 7 for the point arbiter z_0 or the direction arbiter d: its condition is (A), and aligned triples need no condition. The sufficient condition for a point arbiter stated after the theorem is Corollary 2. In both cases the rule is strict, so non-imposition holds as soon as each offered pair is won strictly both ways at some profiles. □

An arbiter placed at the barycentre of μ itself would violate independence, since the tie label of (a, b) would then depend on μ beyond s_{ab}. The corollary avoids this by fixing the barycentre on the domain, which amounts to measuring positions from the centre of the society.

D.5 Separation and offers that react to sizes and means

For $(x, O) \in \mathcal{D}$, let $C_1, \ldots, C_k$ be the cells of O. Condition (S) of Section 4.3 reads: if $\mu_x(C_i) > 0$ for every i, and y is a profile such that μ_y has a finite first moment and, for every i, $\mu_y(C_i) = \mu_x(C_i)$ and the barycentre of μ_y on C_i equals that of μ_x on C_i, then $(y, O) \in \mathcal{D}$.

The mean-of-members mechanism of Example 1 satisfies the part "the offer remains an equilibrium": if O is an equilibrium offer for μ_x, it is one for μ_y. (S) asks in addition that the selection keeps it and that the domain is stable.

Lemma 25 (Spreading inside a cell)**.** *Let μ have a finite first moment, let C be an open convex set with $m = \mu(C) > 0$, and let g be the barycentre of μ on C. Then $g \in C$. For every non-empty bounded open set $\Delta' \subset C$, there is a distribution μ' that coincides with μ outside C, has mass m and barycentre g on C, charges no line, has a finite first moment, and whose support contains Δ'. If μ has compact support, so does μ'.*

Proof. If $g \notin C$, since C is open and convex there is $u \neq 0$ with $\langle u, y\rangle < \langle u, g\rangle$ for every $y \in C$; integrating against $\mu|_C/m$ gives $\langle u, g\rangle < \langle u, g\rangle$. The same argument applies to every probability carried by C with a finite first moment. Let ν be the uniform distribution on Δ' and $h \in C$ its barycentre. Since C is open, for small $s > 0$ the point $q = g + s(g - h)$ lies in C together with a small disc Δ'' around it; let ν' be the uniform distribution on Δ''. With $\lambda = s/(1 + s)$, the barycentre of $\lambda\nu + (1 - \lambda)\nu'$ is $\lambda h + (1 - \lambda)q = g$. For $\beta \in (0, 1)$, let μ' be μ outside C and $(1 - \beta)\mu|_C + \beta m(\lambda\nu + (1 - \lambda)\nu')$ on C. Its mass on C is $(1 - \beta)m + \beta m = m$ and its barycentre on C is g. Uniform distributions on open sets charge no line, so μ' charges no line; ν and ν' have bounded supports, so μ' has a finite first moment, and a compact support if μ has one. Every open set that meets Δ' has positive ν-mass, so the support of μ' contains Δ'. □

Proposition 14 (Separation and (S))**.** *Let every distribution in $\mathcal{D}$ have a finite first moment, and let the mechanism satisfy (S). Let $(x, O) \in \mathcal{D}$ with every cell of positive mass, and let $T \subset O$ be a non-aligned triple. Then there is $(y, O) \in \mathcal{D}$ with $\omega_T \in \operatorname{int} K(\mu_y)$. In particular, a separating mechanism that satisfies (S) and gives every cell a positive mass offers only aligned platforms.*

Proof. Write $\mu = \mu_x$ and let Δ be an open disc centred at ω_T. For each cell C_i that meets Δ, the set $\Delta \cap C_i$ is non-empty, bounded and open, and $\mu(C_i) > 0$. Apply Lemma 25 to these cells in turn, with $\Delta' = \Delta \cap C_i$; each application changes the distribution inside C_i only. The result μ' has a finite first moment, charges no line, and gives every cell the same mass and the same barycentre as μ; so a profile y with distribution μ' satisfies $(y, O) \in \mathcal{D}$ by (S). The support of μ' contains $\Delta \cap C_i$ for every cell that meets Δ, hence all of Δ except points on the bisectors. It is closed, so it contains Δ, and $\omega_T \in \operatorname{int} K(\mu')$. For the last sentence: if some offer contained a non-aligned triple, the offer (y, O) just built would not be separated. □

In words: under (S), each electorate can be spread inside its cell without changing the offer. Citizens can then be placed all around ω_T, which frees the second choices on the triple. A separating mechanism with non-aligned offers must react to the shape of the support, not only to sizes and means.

D.6 The arc mechanism

Definition 6 (Arc mechanism)**.** Fix $\Lambda > 1$ and $\theta \in (0, \frac{\pi}{2})$ with $\Lambda \sin \frac{\theta}{2} < 1$. The domain is the set of all profiles whose distribution has compact support, each with the single offer $O(\mu)$ defined as follows. Let $B(c, \rho)$ be the smallest closed disc that contains the support of μ; it exists and is unique since the support is compact, and $\rho > 0$ since μ charges no line. Let $\omega = c + \Lambda\rho\, e_1$ and $p_j = \omega + \Lambda\rho\,(-\cos\theta_j,\ \sin\theta_j)$ for $\theta_j \in \{-\theta, 0, \theta\}$, so that $p_0 = c$ and $p_\pm = c + \Lambda\rho\,(1 - \cos\theta,\ \pm\sin\theta)$. The offer is $O(\mu) = \{p_-, p_0, p_+\}$: three platforms on an arc of the circle with centre ω and radius $\Lambda\rho$, slightly curved when Λ is large.

The offer depends on μ only, so this is a mechanism in the sense of Section 2. The offer determines (c, ρ): $c = p_0$ and $|p_+ - p_-| = 2\Lambda\rho\sin\theta$.

Proposition 15 (Arc mechanism)**.** *Consider the arc mechanism.*

(i) It is separating, and (A) holds for the direction arbiter $d = -e_1$. The rule M_{-e_1} (majority, ties broken in favour of the platform with the smaller first coordinate, then by $\succ_0$) is a strict total order on every offer.

(ii) M_{-e_1} satisfies non-imposition: each offered pair is won strictly both ways.

(iii) For every $m = (m_-, m_0, m_+)$ in the open simplex and every disc $B(c, \rho)$, some society with smallest disc $B(c, \rho)$ gives masses m to the cells of p_-, p_0, p_+. The set of realized size vectors therefore contains the open simplex. On the subdomain in which every cell has positive mass, it is exactly the open simplex.

(iv) It violates (S).

Proof. Cells. The bisector of c and p_+ passes through ω, since both points lie on the circle with centre ω, and through the point of the arc at angle $\frac{\theta}{2}$; so to the left of ω it lies above the axis $\{y_2 = c_2\}$, which is the bisector of p_- and p_+. All of $B(c, \rho)$ is to the left of ω, since its points have first coordinate at most $c_1 + \rho < c_1 + \Lambda\rho$. The distance from c to this bisector is $|p_+ - c|/2 = \Lambda\rho\sin\frac{\theta}{2} < \rho$, so the bisector cuts the open disc, and the points of the open disc on the side of p_+ are above the axis, hence closer to p_+ than to p_- as well: the cell of

p_+ meets the open disc in a non-empty open set. By symmetry, so does the cell of p_-. The points $c \pm \rho e_1$ lie in the cell of $p_0 = c$: their distance to c is ρ, and

$$\frac{|c + \rho e_1 - p_\pm|^2}{\rho^2} = 1 + 2(1 - \cos\theta)\Lambda(\Lambda - 1) > 1,$$
$$\frac{|c - \rho e_1 - p_\pm|^2}{\rho^2} = 1 + 2(1 - \cos\theta)\Lambda(\Lambda + 1) > 1.$$

(i) The only triple is the offer, and its circumcentre is ω, at distance $\Lambda\rho > \rho$ from c. So $\omega \notin B(c,\rho) \supset K(\mu)$, and the offer is separated. The half-plane $P = \{y_1 < c_1 + \Lambda\rho\}$ is bounded by a line through ω and contains $B(c,\rho)$, so $\mu(P) = 1$; its inward normal is $-e_1$, and $\langle d, -e_1\rangle = 1 > 0$. So (A) holds, and Theorem 7 concludes.

(iii) Let $0 < \eta < m_0/2$. Put mass η on each of two small discs inside $B(c,\rho)$, internally tangent to its circle at $c \pm \rho e_1$ and contained in the cell of p_0, which is open and contains these points. Put mass $m_0 - 2\eta$ on a small disc centred at c, and masses m_- and m_+ on small discs inside $B(c,\rho)$ and inside the cells of p_- and p_+. The support lies in $B(c,\rho)$ and contains the two diametrically opposite points $c \pm \rho e_1$. A closed disc that contains two points at distance 2ρ has radius at least ρ, with equality only for the disc with that diameter. So the smallest disc is $B(c,\rho)$, the offer is $\{p_-, p_0, p_+\}$, and the masses of the cells are m. On the subdomain with cells of positive mass, the sizes are positive and sum to 1, since the bisectors are null; so they lie in the open simplex, and by the construction they fill it.

(ii) The offer determines (c,ρ). Use the construction of (iii) for this (c,ρ). With m_0 close to 1, the members of p_0 form a majority and $s_{p_0 p_\pm} > \frac{1}{2}$. With m_+ close to 1, $s_{p_0 p_+} < \frac{1}{2}$ and $s_{p_+ p_-} > \frac{1}{2}$; with m_- close to 1, $s_{p_0 p_-} < \frac{1}{2}$ and $s_{p_- p_+} > \frac{1}{2}$. M_{-e_1} follows strict majority off ties, so each pair is won strictly both ways.

(iv) Take μ from (iii) and let C_0 be the cell of p_0, of mass $m_0 > 0$. It contains every point $c - te_1$, $t > 0$, since $|c - te_1 - p_\pm|^2 = (t + \Lambda\rho(1 - \cos\theta))^2 + \Lambda^2\rho^2 \sin^2\theta > t^2 = |c - te_1 - p_0|^2$. Choose $R > 4\rho$ and an open disc $\Delta' \subset C_0$ of radius $r < \rho$ centred at $c - Re_1$. Lemma 25 gives μ', equal to μ outside C_0, with the same mass and barycentre as μ on C_0, compact support, no mass on lines, and a support that contains Δ'. So μ' is in the domain, with the same sizes and means as μ on the three cells of $O(\mu)$. Its support contains points of the two other cells, within ρ of c, and points of Δ', at distance at least $R - r$ from c; so its smallest disc has radius at least $(R - r - \rho)/2 > \rho$. The offer determines this radius, so $O(\mu') \neq O(\mu)$, and (S) fails, even among compact supports. □

Remarks S13 and S14 in the Supplementary Appendix describe the size vectors realized when empty parties are allowed, and extend the mechanism to distributions with unbounded support, where it offers three aligned platforms.

References

J. H. Aldrich, *Why Parties?*, University of Chicago Press, 1995.

K. J. Arrow, A difficulty in the concept of social welfare, *Journal of Political Economy* 58 (1950), 328–346.

K. J. Arrow, *Social Choice and Individual Values*, Wiley, 1951.

S. Barbaro and A.-S. Kurella, On the prevalence of Condorcet's paradox, *Public Choice* 208 (2026), 29–47.

T. Besley and S. Coate, An economic model of representative democracy, *Quarterly Journal of Economics* 112 (1997), 85–114.

D. Black, On the rationale of group decision-making, *Journal of Political Economy* 56 (1948), 23–34.

N. Boissonnet, How large are majority cycles among parties?, working paper, 2026.

K. C. Border, Social welfare functions for economic environments with and without the Pareto principle, *Journal of Economic Theory* 29 (1983), 205–216.

K. C. Border, An impossibility theorem for spatial models, *Public Choice* 43 (1984), 293–305.

A. Caplin and B. Nalebuff, On 64%-majority rule, *Econometrica* 56 (1988), 787–814.

A. Caplin and B. Nalebuff, Competition among institutions, *Journal of Economic Theory* 72 (1997), 306–342.

G. W. Cox, *Making Votes Count: Strategic Coordination in the World's Electoral Systems*, Cambridge University Press, 1997.

P. Dasgupta and E. Maskin, On the robustness of majority rule and unanimity rule, working paper, Institute for Advanced Study, Princeton, version of October 2003.

P. Dasgupta and E. Maskin, On the robustness of majority rule, *Journal of the European Economic Association* 6 (2008), 949–973.

V. Denicolò, Independent social choice correspondences are dictatorial, *Economics Letters* 19 (1985), 9–12.

Z. Desai and T. Kalandrakis, The core of the party system, *Journal of Politics* 87 (2025), 1499–1514.

F. Dietrich and C. List, Judgment aggregation by quota rules: majority voting generalized, *Journal of Theoretical Politics* 19 (2007), 391–424.

J. S. Dryzek and C. List, Social choice theory and deliberative democracy: a reconciliation, *British Journal of Political Science* 33 (2003), 1–28.

B. Dutta, M. O. Jackson and M. Le Breton, Strategic candidacy and voting procedures, *Econometrica* 69 (2001), 1013–1037.

M. Duverger, *Political Parties: Their Organization and Activity in the Modern State*, Methuen, 1954 (French original: *Les partis politiques*, Armand Colin, 1951).

L. Ehlers and T. Storcken, Arrow's possibility theorem for one-dimensional single-peaked preferences, *Games and Economic Behavior* 64 (2008), 533–547.

T. J. Feddersen, I. Sened and S. G. Wright, Rational voting and candidate entry under plurality rule, *American Journal of Political Science* 34 (1990), 1005–1016.

A. Gomberg, F. Marhuenda and I. Ortuño-Ortín, A model of endogenous political party platforms, *Economic Theory* 24 (2004), 373–394.

A. Gomberg, C. Martinelli and R. Torres, Anonymity in large societies, *Social Choice and Welfare* 25 (2005), 187–205.

J. Greenberg, Consistent majority rules over compact sets of alternatives, *Econometrica* 47 (1979), 627–636.

D. M. Grether and C. R. Plott, Nonbinary social choice: an impossibility theorem, *Review of Economic Studies* 49 (1982), 143–149.

M. Grötschel, M. Jünger and G. Reinelt, Facets of the linear ordering polytope, *Mathematical Programming* 33 (1985), 43–60.

P. R. Halmos and J. von Neumann, Operator methods in classical mechanics, II, *Annals of Mathematics* 43 (1942), 332–350.

K. Inada, The simple majority decision rule, *Econometrica* 37 (1969), 490–506.

Y. Iwata, The possibility of Arrovian social choice with the process of nomination, *Theory and Decision* 81 (2016), 535–552.

E. Kalai and E. Muller, Characterization of domains admitting nondictatorial social welfare functions and nonmanipulable voting procedures, *Journal of Economic Theory* 16 (1977), 457–469.

E. Kalai, E. Muller and M. A. Satterthwaite, Social welfare functions when preferences are convex, strictly monotonic, and continuous, *Public Choice* 34 (1979), 87–97.

A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

R. Lachat and J.-F. Laslier, Alternatives to plurality rule for single-winner elections: when do they make a difference?, *European Journal of Political Economy* 81 (2024), 102505.

M. Laver and E. Sergenti, *Party Competition: An Agent-Based Model*, Princeton University Press, 2012.

M. Le Breton and J. A. Weymark, Arrovian social choice theory on economic domains, in K. J. Arrow, A. K. Sen and K. Suzumura (eds.), *Handbook of Social Choice and Welfare*, vol. 2, North-Holland, 2011, ch. 17, 191–299.

G. Levy, A model of political parties, *Journal of Economic Theory* 115 (2004), 250–277.

C. List, R. C. Luskin, J. S. Fishkin and I. McLean, Deliberation, single-peakedness, and the possibility of meaningful democracy: evidence from deliberative polls, *Journal of Politics* 75 (2013), 80–95.

G. Mackie, *Democracy Defended*, Cambridge University Press, 2003.

K. O. May, A set of independent necessary and sufficient conditions for simple majority decision, *Econometrica* 20 (1952), 680–684.

R. D. McKelvey, Intransitivities in multidimensional voting models and some implications for agenda control, *Journal of Economic Theory* 12 (1976), 472–482.

M. Morelli, Party formation and policy outcomes under different electoral systems, *Review of Economic Studies* 71 (2004), 829–853.

H. Moulin, On strategy-proofness and single peakedness, *Public Choice* 35 (1980), 437–455.

H. Moulin, Generalized Condorcet-winners for single peaked and single-plateau preferences, *Social Choice and Welfare* 1 (1984), 127–147.

K. Nehring and C. Puppe, The structure of strategy-proof social choice. Part I: general characterization and possibility results on median spaces, *Journal of Economic Theory* 135 (2007), 269–305.

K. Nehring and C. Puppe, Abstract Arrowian aggregation, *Journal of Economic Theory* 145 (2010), 467–494.

M. J. Osborne and A. Slivinski, A model of political competition with citizen-candidates, *Quarterly Journal of Economics* 111 (1996), 65–96.

C. R. Plott, A notion of equilibrium and its possibility under majority rule, *American Economic Review* 57 (1967), 787–806.

C. Puppe and A. Slinko, *Condorcet Domains: The Mathematics of Coherent Collective Decision-Making*, Springer, 2026.

W. H. Riker, *Liberalism against Populism*, Freeman, 1982.

R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

A. K. Sen, A possibility theorem on majority decisions, *Econometrica* 34 (1966), 491–499.

A. K. Sen, The possibility of social choice, *American Economic Review* 89 (1999), 349–378.

A. K. Sen and P. K. Pattanaik, Necessary and sufficient conditions for rational choice under majority decision, *Journal of Economic Theory* 1 (1969), 178–202.

J. Sethuraman, C.-P. Teo and R. V. Vohra, Anonymous monotonic social welfare functions, *Journal of Economic Theory* 128 (2006), 232–254.

G. Tullock, The general irrelevance of the general impossibility theorem, *Quarterly Journal of Economics* 81 (1967), 256–270.

B. Ward, Majority voting and alternative forms of public enterprises, in J. Margolis (ed.), *The Public Economy of Urban Communities*, Johns Hopkins Press, 1965, 112–126.

R. Wilson, Social choice theory without the Pareto principle, *Journal of Economic Theory* 5 (1972), 478–486.