Supplementary Appendix to "Do Parties Solve Arrow's Problem?"

Niels Boissonnet*

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S1 Robustness

This appendix proves the first three variants listed in Section 8 of the paper (distributions that charge lines, selections among equilibria, and offers whose parties are crowded together), and treats offers whose triples are collinear; the fourth variant of Section 8 is proved in Appendix S4 below. Each subsection states what is proved and what is open.

Throughout, a, b, c are platforms, $T = \{a, b, c\}$, shares on T are written in cyclic order, and h is the centre, the point where every duel is tied. $\mathcal{M}_0$ is the set of probability distributions on $\mathbb{R}^2$ that charge no line, and $\mathcal{M}_1$ is the set of finite mixtures of uniform distributions on discs and on segments of the plane. Distributions of $\mathcal{M}_1$ charge lines but no point. Every probability distribution on $\mathbb{R}^2$ is the distribution μ_x of some profile, since I is a standard non-atomic space; so a domain may be described by a set of distributions and a set of offers.

S1.1 Ties of positive mass

Vector shares. When μ_x may charge a line, a positive mass of citizens may be indifferent between two platforms, and $s_{ab} + s_{ba}$ may be less than 1. The share of the pair is then the vector

$$\mathbf{s}_{ab} = (s_{ab}, s_{ba}) = (\mu_x(H_{ab}), \mu_x(H_{ba})) \in \Theta = \{(p, q) : \ p, q \geq 0, \ p + q \leq 1\}.$$

The mass $1 - s_{ab} - s_{ba}$ is the mass of the citizens indifferent between a and b. I also use $\Theta_\varepsilon = \{(p, q) : \ p, q \geq \varepsilon, \ p + q \leq 1\}$ and $\Theta^\circ = \{(p, q) : \ p, q > 0, \ p + q \leq 1\}$.

With ties, I read Axiom 2 with both sets: $F(x, O)$ and $F(y, O')$ rank a and b in the same way when the sets $\{i : a \succ_i b\}$ coincide up to a null set and the sets $\{i : b \succ_i a\}$ also coincide. This is the weaker reading of the axiom, so impossibility results are stronger under it; the possibility examples below satisfy it. By Lemma 4, a rule that satisfies anonymity and independence is a labelling: for each ordered pair (a, b) there is a map ℓ_{ab} from the set $D_{ab} \subset \Theta$ of producible vector shares to $\{\mathsf{F}, \mathsf{S}, \mathsf{I}\}$, and ℓ_{ab} depends on the pair and on nothing else. On a triple, the share vector is $(\mathbf{s}_{ab}, \mathbf{s}_{bc}, \mathbf{s}_{ca})$, and V_T is the set of such vectors over all (x, O) with $T \subset O$. Validity on a triple is defined as in Section 3. Non-imposition for the pair says that ℓ_{ab} is neither constantly F nor constantly S on D_{ab}; this is Axiom 3 for the two orders of the pair.

Remark S1 (The star lemma holds verbatim). Lemmas 5, 2 and 3 look only at labels. They never use that the coordinates of a share vector are numbers. They therefore hold word for word with $D_i \subset \Theta$, with products $X \times Y \times Z \subset V_T$ of subsets of Θ, and with a common point

*References to numbered results without the prefix S are to the paper.

$h \in V_T$. So does Theorem 1, on any domain that satisfies Assumption 1, once star triples are defined with vector shares.

The proofs below use a special case: products in which two factors are single points.

Lemma S1 (Cross)**.** *Let V be the set of share vectors produced on a triple, $h = (h_1, h_2, h_3) \in V$, and J_1, J_2, J_3 sets with $h_i \in J_i$. Suppose that, for each coordinate i and each $w \in J_i$, the point obtained by replacing h_i by w lies in V. Let a rule be valid on V, and suppose that for each i the label ℓ_i is neither constantly* F *nor constantly* S *on J_i. Then $\ell_i =$* I *on all of J_i, for $i = 1, 2, 3$.*

Proof. Let $e_i = \ell_i(h_i)$.

Suppose first $(e_1, e_2, e_3) = (\mathsf{I}, \mathsf{I}, \mathsf{I})$. For $w \in J_1$ the point (w, h_2, h_3) carries the labels $(\ell_1(w), \mathsf{I}, \mathsf{I})$. A valid triple with two I is $(\mathsf{I}, \mathsf{I}, \mathsf{I})$, since it needs either no strict label or both an F and an S. So $\ell_1(w) = \mathsf{I}$. The same holds for coordinates 2 and 3.

Otherwise the labels at h contain an F and an S. Let i be a coordinate with e_i strict, and j, k the two others. I claim that e_j, e_k are strict and opposite. If not, $\{e_j, e_k\}$ misses one of the two strict labels; say it misses S, the other case being symmetric. Since (e_i, e_j, e_k) contains an S, $e_i =$ S. By hypothesis there is $w \in J_i$ with $\ell_i(w) \neq$ S. The point h modified at i carries $(\ell_i(w), e_j, e_k)$, with no S; for validity it is $(\mathsf{I}, \mathsf{I}, \mathsf{I})$. Then $e_j = e_k =$ I, and $(e_i, e_j, e_k) =$ (S, I, I) is not valid: a contradiction.

So for every strict coordinate at h, the two others are strict and opposite. This is impossible. If some label at h is I, a strict coordinate sees that I among the two others. If all three are strict, two are equal, and the third sees them. □

Remark S2 (Crosses and stars). With $J_i = D_i$, Lemma S1 is the star lemma for the family of products $\{h_1, w\} \times \{h_2\} \times \{h_3\}$ and their rotations. Conversely, a family of products through h whose factors cover D_1, D_2, D_3 contains every point h modified at i by some $w \in D_i$. So a triple is a star triple if and only if there is $h \in V_T$ such that V_T contains the three crosses through h with arms D_{ab}, D_{bc}, D_{ca}. It suffices to produce share vectors of the form (w, h_2, h_3) and their rotations; whole boxes are not needed.

Thirteen types. Let T be non-aligned, with circumcentre ω. The three bisectors meet at ω and cut the plane into thirteen regions, which give thirteen *types* of preferences on T. There are six open sectors, one per strict order (abc, acb, and so on; Lemma 6). There are six open half-lines from ω. On $L_{ab} \setminus \{\omega\}$, the difference $|z - c| - |z - a|$ vanishes only at ω, so it has a fixed sign on each of the two half-lines. On one of them, written $[c|ab]$, c is strictly first and $a \sim b$; it is the ray r_c of Lemma 6, inside the cell of c relative to T. On the other, written $[ab|c]$, $a \sim b$ and c is strictly last; it is the common boundary of the cells of a and b. The half-lines $[a|bc]$, $[bc|a]$ on L_{bc} and $[b|ca]$, $[ca|b]$ on L_{ca} are defined in the same way. The last type is the point ω, where all three platforms are tied. Shares are linear in the masses of the types; for instance

$$s_{ab} = m(abc) + m(acb) + m(cab) + m([a|bc]) + m([ca|b]).$$

A family of masses on the types is produced by a distribution of $\mathcal{M}_1$ as soon as ω gets no mass: put each mass uniformly on a small disc inside its sector, or on a small segment inside its half-line.

Lemma S2 (Half first, half last)**.** *Let T be non-aligned and $(p, q) \in \Theta$. There is $\mu \in \mathcal{M}_1$ with*

$$\mathbf{s}_{ab}(\mu) = (p, q), \qquad \mathbf{s}_{bc}(\mu) = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad \mathbf{s}_{ca}(\mu) = (\tfrac{1}{2}, \tfrac{1}{2}).$$

If $p + q = 1$, one can take $\mu \in \mathcal{M}_0$. The same holds after rotating the names $a \to b \to c \to a$.

Proof. Put the mass $\frac{1}{2}$ on the types where c is strictly first: cab (mass α), cba (mass β), $[c|ab]$ (mass $\frac{1}{2}-\alpha-\beta$). Put the mass $\frac{1}{2}$ on the types where c is strictly last: abc (mass α'), bac (mass β'), $[ab|c]$ (mass $\frac{1}{2}-\alpha'-\beta'$). The first group prefers c to a and to b; the second prefers a and b to c. So $s_{bc}=s_{cb}=s_{ca}=s_{ac}=\frac{1}{2}$, whatever $\alpha,\beta,\alpha',\beta'$. And $\mathbf{s}_{ab}=(\alpha+\alpha',\beta+\beta')$. When (α,β) and (α',β') range over $\frac{1}{2}\Theta$, their sum ranges over $\frac{1}{2}\Theta+\frac{1}{2}\Theta=\Theta$. If $p+q=1$, the masses on the two half-lines are zero and the distribution charges no line. □

In words: half of the citizens put c first, the other half put c last, and each citizen thinks what she likes of a against b, indifference included. The duels (b,c) and (c,a) stay exactly tied while the duel (a,b) takes every value.

Theorem S1 (Fixed offer, with or without ties)**.** *Let S be a fixed offer with $|S|\geq 3$ that is not aligned: its points do not all lie on one line. Let $\mathcal{M}=\mathcal{M}_0$, or $\mathcal{M}\supseteq\mathcal{M}_1$. On the domain $\{(x,S):\mu_x\in\mathcal{M}\}$, no rule satisfies anonymity, independence (within the offer suffices), non-imposition and non-nullness.*

Proof. Each pair lies in a non-aligned triple: for $a\neq b$ in S, S is not contained in the line (ab), so some $c\in S$ lies off that line. Fix such a triple T, and let $h=((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$.

If $\mathcal{M}=\mathcal{M}_0$, the produced shares of a pair are $D_{ab}=\{(p,1-p):p\in[0,1]\}$, all produced by Lemma S2. If $\mathcal{M}\supseteq\mathcal{M}_1$, $D_{ab}\subset\Theta$, and all of Θ is produced by $\mathcal{M}_1$; so $D_{ab}=\Theta$. In both cases Lemma S2 and its rotations give the hypothesis of Lemma S1 with $J_i=D_i$, and $h\in V_T$.

By Lemma 4 and Remark 1, the rule is a labelling. The restriction of $F(x,S)$ to T is a weak order, so the rule is valid on T. Non-imposition bears on the sets D_{ab}. By Lemma S1, the pair $\{a,b\}$ is indifferent everywhere. This holds for every pair of S: the rule is null. □

For $\mathcal{M}=\mathcal{M}_0$ this is the impossibility for a fixed offer under the assumption of the paper, with a proof of a few lines and the weak hypothesis "not aligned". For larger domains the cross also passes through the centre, with indifferent citizens on its arms.

Impossibility is not monotone in the domain. The hypothesis $\mathcal{M}\supseteq\mathcal{M}_1$ is not there for convenience. Between $\mathcal{M}_0$ and the set of all distributions, some domains admit a rule.

Example S1 (Adding one distribution)**.** Let $S=\{a,b,c\}$ be non-aligned, with circumcentre ω, and $\mathcal{M}=\mathcal{M}_0\cup\{\delta_\omega\}$. The rule "on $\mathcal{M}_0$, a is strictly first and $b\sim c$; at δ_ω, all are indifferent" satisfies anonymity, independence, non-imposition and non-nullness.

Proof. On $\mathcal{M}_0$ the vector shares lie on the hypotenuse $p+q=1$; at δ_ω every citizen is indifferent between every two platforms, and the vector shares are $(0,0)$ for the three pairs. The labels are therefore functions of the vector shares of their pair, which gives anonymity and independence. Each ranking is a weak order. Non-imposition: $a\succ b$ and $a\succ c$ on $\mathcal{M}_0$; $b\sim a$ and $c\sim a$ at δ_ω; $b\sim c$ everywhere. Non-nullness: $a\succ b$ on $\mathcal{M}_0$. □

Example S2 (Second-rank indifference only)**.** Let $S=\{a,b,c\}$ be non-aligned, and let $\mathcal{M}'$ be the set of distributions that charge neither ω nor the three half-lines $[ab|c]$, $[bc|a]$, $[ca|b]$. So $\mathcal{M}_0\subset\mathcal{M}'$: every citizen has a strict first choice, but may be indifferent between her two other platforms. The following rule satisfies anonymity, independence, non-imposition and non-nullness on $\mathcal{M}'$:

- $a\succ b$ if $s_{ab}+s_{ba}=1$, and $a\sim b$ otherwise;
- $c\succ b$ if $s_{bc}+s_{cb}>0$, and $b\sim c$ otherwise;
- $c\succ a$ if $s_{ca}>0$, and $a\succ c$ if $s_{ca}=0$.

Proof. Each label depends only on the vector share of its pair, which gives anonymity and independence. Let $\mu \in \mathcal{M}'$ and let m_c be the mass of the citizens who put c strictly first.

If $m_c > 0$, these citizens prefer c to b and to a: $s_{cb} > 0$ and $s_{ca} > 0$, so $c \succ b$ and $c \succ a$. Whatever the label of $\{a, b\}$, the result is a weak order with c first.

If $m_c = 0$, almost every citizen puts a or b strictly first, since ω and the outer half-lines are not charged. No citizen is then indifferent between a and b, and $a \succ b$. If $s_{bc} + s_{cb} > 0$, then $c \succ b$; with $a \succ b$, b is last and the result is a weak order whatever the label of $\{c, a\}$. Otherwise $s_{bc} = s_{cb} = 0$: every citizen is indifferent between b and c. The citizens who put b first prefer b to c, so there are none. All put a first, so $s_{ca} = 0$ and $a \succ c$. With $a \succ b$ and $b \sim c$, this is a weak order.

Non-imposition: $a \succ b$ on $\mathcal{M}_0$; $a \sim b$ for a distribution carried by $[c|ab]$; $c \succ b$ on $\mathcal{M}_0$ when c has members; $b \sim c$ for a distribution carried by $[a|bc]$; $c \succ a$ on $\mathcal{M}_0$; $a \succ c$ for a distribution carried by the sector abc. All these distributions lie in $\mathcal{M}'$. Non-nullness is clear. □

On $\mathcal{M}_0$ and on every $\mathcal{M} \supseteq \mathcal{M}_1$ there is no rule; on $\mathcal{M}_0 \cup \{\delta_\omega\}$ and on $\mathcal{M}'$ an odd rule exists. Wilson's non-imposition can be met by a few extreme distributions, and these must then be "seen" from the centre, which is what the cross guarantees. What $\mathcal{M}'$ lacks are the citizens whose first two choices are tied. Without them, very indifferent values such as $s_{ab} + s_{ba} < \frac{1}{2}$ require more than half of the citizens to put c first, and the cross no longer passes through the centre.

Free arbitration with ties

On a triple, suppose that every citizen has a strict first choice, with sizes $m = (m_a, m_b, m_c)$, $\sum m = 1$, and that each member of c may prefer a to b, prefer b to a, or be indifferent; likewise for the other parties. At fixed sizes, the produced share vectors form the product $\tilde{B}(m) = X(m) \times Y(m) \times Z(m)$, where

$$X(m) = \{(p, q) : p \geq m_a,\ q \geq m_b,\ p+q \leq 1\}, \quad Y(m) = \{(p, q) : p \geq m_b,\ q \geq m_c,\ p+q \leq 1\},$$

$$Z(m) = \{(p, q) : p \geq m_c,\ q \geq m_a,\ p + q \leq 1\}.$$

Indeed $(p - m_a) + (q - m_b) \leq m_c$ is the same as $p + q \leq 1$. For a non-empty set M of sizes, let $V = \bigcup_{m \in M} \tilde{B}(m)$ and let D_1, D_2, D_3 be its projections.

Theorem S2 (Free arbitration with ties)**.** *Let M be a non-empty set of sizes and $\bar{m}_i = \sup_{m \in M} m_i$ for $i \in \{a, b, c\}$. The following are equivalent.*

(i) Every rule that is valid on V and non-imposed is null.

(ii) The products $\tilde{B}(m)$, $m \in M$, have a common point.

(iii) $\bar{m}_i + \bar{m}_j \leq 1$ for all $i \neq j$.

Proof. (ii)⇒(iii). If $h = (h_1, h_2, h_3)$ lies in every $\tilde{B}(m)$, then $h_1 = (p, q)$ satisfies $p \geq m_a$ and $q \geq m_b$ for every $m \in M$. So $\bar{m}_a + \bar{m}_b \leq p + q \leq 1$. The same holds with h_2 and h_3.

(iii)⇒(ii). Let $h_1 = (\bar{m}_a, 1 - \bar{m}_a)$, $h_2 = (\bar{m}_b, 1 - \bar{m}_b)$, $h_3 = (\bar{m}_c, 1 - \bar{m}_c)$. For $m \in M$: $h_1 \in X(m)$, since $\bar{m}_a \geq m_a$ and $1 - \bar{m}_a \geq \bar{m}_b \geq m_b$; $h_2 \in Y(m)$, since $\bar{m}_b \geq m_b$ and $1 - \bar{m}_b \geq \bar{m}_c \geq m_c$; $h_3 \in Z(m)$, since $\bar{m}_c \geq m_c$ and $1 - \bar{m}_c \geq \bar{m}_a \geq m_a$.

(ii)⇒(i). The products $\tilde{B}(m)$ lie in V, all contain h, and their factors cover D_1, D_2, D_3 by definition. The star lemma (Remark S1) concludes.

Not (iii) ⇒ not (i). Suppose, say, $\bar{m}_a + \bar{m}_b > 1$; the other pairs of parties are treated by renaming. There are $m^1, m^2 \in M$ with $m^1_a + m^2_b > 1$. Let

$$w_2 = (m^1_b, m^1_c) \in D_2, \qquad w_3 = (m^2_c, m^2_a) \in D_3,$$

$$M_A = \{m \in M :\ m_b \le m_b^1,\ m_c \le m_c^1\}, \qquad M_B = \{m \in M :\ m_c \le m_c^2,\ m_a \le m_a^2\}.$$

Then $w_2 \in Y(m)$ if and only if $m \in M_A$, and $w_3 \in Z(m)$ if and only if $m \in M_B$. On M_A, $m_a = 1 - m_b - m_c \ge m_a^1$; on M_B, $m_b \ge m_b^2$. So $M_A \cap M_B = \emptyset$, since otherwise $m_a + m_b > 1$. Let $X_A = \bigcup_{m \in M_A} X(m)$, whose points have $p \ge m_a^1$, and $X_B = \bigcup_{m \in M_B} X(m)$, whose points have $q \ge m_b^2$. So $X_A \cap X_B = \emptyset$, since otherwise $p + q > 1$. Define the rule:

- $\ell_{bc} = \mathsf{F}$ $(b \succ c)$ at w_2, and S $(c \succ b)$ elsewhere;
- $\ell_{ca} = \mathsf{S}$ $(a \succ c)$ at w_3, and F $(c \succ a)$ elsewhere;
- $\ell_{ab} = \mathsf{S}$ $(b \succ a)$ on X_A, and F $(a \succ b)$ elsewhere.

Validity. Let $m \in M$. If $m \notin M_A \cup M_B$, then $w_2 \notin Y(m)$ and $w_3 \notin Z(m)$, so $c \succ b$ and $c \succ a$ on the whole product: c is first, and the result is a weak order. If $m \in M_A$, then $m \notin M_B$, so $c \succ a$ on $Z(m)$; and $X(m) \subset X_A$, so $b \succ a$: a is last. If $m \in M_B$, then $m \notin M_A$, so $c \succ b$ on $Y(m)$; and $X(m) \subset X_B$, which is disjoint from X_A, so $a \succ b$: b is last.

Non-imposition. ℓ_{ab} is S on $X(m^1) \subset X_A$ and F on $X(m^2) \subset X_B$. ℓ_{bc} is F at w_2 and S at $(m_b^1 + m_a^1, m_c^1) \in Y(m^1)$, which differs from w_2 since $m_a^1 > 1 - m_b^2 \ge 0$. Likewise ℓ_{ca} is S at w_3 and F at $(m_c^2 + m_b^2, m_a^2) \ne w_3$. Non-nullness is clear. Each label depends only on the vector share of its pair. □

Corollary S1 (Minimum sizes and symmetric sizes)**.** *(a) Let $M_\varepsilon = \{m :\ m_a, m_b, m_c \ge \varepsilon\}$ with $0 \le \varepsilon \le \frac{1}{3}$. With ties, every valid, non-imposed rule is null if and only if $\varepsilon \ge \frac{1}{4}$. (b) If M is invariant under the permutations of the three parties, every valid, non-imposed rule is null if and only if $M \subset \{\max m \le \frac{1}{2}\}$: no party ever holds a majority.*

Proof. (a) $\bar{m}_i = 1 - 2\varepsilon$ for each i, and $2(1 - 2\varepsilon) \le 1$ is $\varepsilon \ge \frac{1}{4}$. (b) The three $\bar{m}_i$ equal $\bar{m} = \sup_M \max m$, and (iii) reads $2\bar{m} \le 1$. □

Take $\varepsilon = 0.1$, $m^1 = (0.8, 0.1, 0.1)$ and $m^2 = (0.1, 0.8, 0.1)$. Then $M_A = \{m^1\}$, $M_B = \{m^2\}$ and $w_2 = w_3 = (0.1, 0.1)$. The rule says that c beats a and b, except in two extreme situations. When the duel (b, c) is $(0.1, 0.1)$, a has 80 per cent of the members, all indifferent between b and c; the rule puts $b \succ c$, and a last. When the duel (c, a) is $(0.1, 0.1)$, b has 80 per cent of the members, all indifferent between c and a; the rule puts $a \succ c$, and b last. The duel (a, b) follows: $b \succ a$ if $s_{ab} \ge 0.8$, and $a \succ b$ otherwise.

The mechanism is the following. With ties, the value (m_b, m_c) of the duel (b, c), where all members of a are indifferent, is produced only by the sizes at which b and c are at most as large as in m. It is a corner of $Y(m)$. For a star to pass through that corner, it must meet every product that contains the corner, and, letting the corner vary, every product: this is (ii). If two parties can be larger together than the whole, in two different configurations, the two corresponding corners are seen by disjoint sets of sizes, with disjoint values of the third duel. The rule can then be reversed at each corner without creating an inconsistency.

Without ties, the recipe of Corollary 1 only asks for *some* products through h that cover the shares, and M_ε meets it for every $\varepsilon \le \frac{1}{3}$. With ties, *all* products must have a common point, and the threshold moves from 0 to $\frac{1}{4}$. The common point, when it exists, need not be the centre: for $M = \{(0.7, 0.3, 0), (0.7, 0, 0.3), (0.4, 0.3, 0.3)\}$, $\bar{m} = (0.7, 0.3, 0.3)$ satisfies (iii), and the impossibility holds although a sometimes holds a majority. The rule in the proof is non-imposed only at isolated points, w_2 and w_3. Whether a rule with weak robust non-imposition exists for M_ε, $\varepsilon < \frac{1}{4}$, is open.

Minimum size with ties

Definition S1 (Domain)**.** Let $k \geq 3$, $0 < \varepsilon \leq 1/k$, and $\mathcal{M} \supseteq \mathcal{M}_1$ a set of distributions. The domain $\mathcal{D}^{\text{tie}}_{k,\varepsilon}(\mathcal{M})$ is the set of pairs (x, O) such that $\mu_x \in \mathcal{M}$, O is any set of k distinct points of the plane, and each *open* cell of O has mass at least ε. Citizens on the boundaries between cells are members of no party.

For instance, $\mathcal{M}$ may be the set of all distributions, or of all distributions without atoms. The domain satisfies Assumption 1.

Lemma S3 (Shares to cover)**.** *For every offered pair, $D_{ab} \subset \Theta_\varepsilon$.*

Proof. In every offer $O \ni a, b$, the open cell of a lies in H_{ab} and has mass at least ε, so $s_{ab} \geq \varepsilon$. Likewise $s_{ba} \geq \varepsilon$. □

Fix a pair $\{a, b\}$ and a point c off the line (ab); let $T = \{a, b, c\}$, ω its circumcentre and R_T its circumradius. Let u be the unit vector of the half-line $r_c = [c|ab]$; the half-line $[ab|c]$ has direction $-u$. Let $j_c, j' \geq 0$ be integers with $j_c + j' = k - 3$, chosen below. For large $\tau_0 > 0$, place j_c points d at $\omega + \tau u$ and j' points d at $\omega - \tau u$, with values $\tau \geq \tau_0$ that are distinct and pairwise at least 1 apart. The offer is $O = T \cup \{d\}$, with k points. The far points are said to lie *on* r_c or *on* $[ab|c]$.

Lemma S4 (Placements)**.** *Let $R_0 > 0$. For τ_0 large enough:*

(i) every point z within R_0 of ω that is interior to the cell of $t \in T$ relative to T lies in the open cell of t relative to O; every point of $[ab|c]$ within R_0 of ω lies on the common boundary of the cells of a and b relative to O;

(ii) for a far point d, the ball $B(d, 1/4)$ lies in the open cell of d relative to O;

(iii) if d lies on r_c, $B(d, 1/4)$ lies in the cell of c relative to T, and L_{ab} cuts it into two halves, one in the sector cab and the other in cba; if d lies on $[ab|c]$, $B(d, 1/4)$ lies in the region where c is strictly last, and L_{ab} cuts it into a half in abc and a half in bac.

Proof. (i) If $|z - \omega| \leq R_0$, then $|z - t| \leq R_0 + R_T$ for $t \in T$, and $|z - d| \geq \tau_0 - R_0$ for every far point. For $\tau_0 > 2R_0 + R_T$, the far points are strictly farther than the three points of T, so the cell of z relative to O is its cell relative to T. A point equidistant from a and b, and closer to them than to c, stays on the boundary between a and b.

(ii) Let $z \in B(d, 1/4)$. Another far point on the same side is at distance at least $1 - 1/4$ from z; a far point on the opposite side is at distance at least $2\tau_0 - 1/4$; a point $t \in T$ is at distance at least $\tau_0 - R_T - 1/4$. For τ_0 large, all exceed $1/4 > |z - d|$.

(iii) The cell of c relative to T is the intersection of two open half-planes whose boundaries pass through ω, and $\omega + \tau u$ lies in both for $\tau > 0$. The distance from $\omega + \tau u$ to each boundary is τ times that of $\omega + u$, and exceeds $1/4$ for τ large. The same argument applies to the region $\{|z - c| > |z - a|\} \cap \{|z - c| > |z - b|\}$, which contains $\omega - \tau u$ for $\tau > 0$. Finally $d \in L_{ab}$, and L_{ab} separates the two sectors in question. □

Supports. Fix a finite family $\mathcal{S}$ of supports, each inside one type of T and inside one open cell of O, or on the boundary between a and b. Near ω, within R_0: a small disc in each of the six sectors, a segment on r_c, and a segment on $[ab|c]$. For each far point d: two small discs in $B(d, 1/4)$, one on each side of L_{ab}, and a segment of L_{ab} inside $B(d, 1/4)$. By Lemma S4, each party can host the following types.

party	types hosted
a (cell near ω)	abc, acb
b	bac, bca
c	cab, cba, $[c\|ab]$
far point on r_c	cab, cba, $[c\|ab]$
far point on $[ab\|c]$	abc, bac, $[ab\|c]$
none (boundary near ω)	$[ab\|c]$

A distribution is a weight vector $w \in \mathbb{R}_+^{\mathcal{S}}$ of sum 1, uniform on each support; it lies in $\mathcal{M}_1$. Cell masses and shares are linear in w.

Let

$$\text{for } k \text{ odd: } t = \tfrac{1}{2},\ j_c = j' = \tfrac{k-3}{2}; \qquad \text{for } k \text{ even: } t = 1,\ j_c = \tfrac{k-4}{2},\ j' = \tfrac{k-2}{2}.$$

Lemma S5 (Two conditions)**.** *With these choices and $k\varepsilon \le 1$,*

$$\textit{(C1)} \quad \tfrac{1}{2} - t\varepsilon \ge (1 + j_c)\varepsilon, \qquad\qquad \textit{(C2)} \quad \tfrac{1}{2} - 2\varepsilon + t\varepsilon \ge j'\varepsilon.$$

Proof. (C1) reads $1 + j_c + t \le \frac{1}{2\varepsilon}$ and (C2) reads $j' + 2 - t \le \frac{1}{2\varepsilon}$. For k odd both left-hand sides equal $\frac{k-1}{2} + \frac{1}{2} = \frac{k}{2}$. For k even they equal $\frac{k-2}{2} + 1 = \frac{k}{2}$. And $\frac{k}{2} \le \frac{1}{2\varepsilon}$. □

Lemma S6 (Vertices)**.** *For each of the three vertices $(\varepsilon, \varepsilon)$, $(1-\varepsilon, \varepsilon)$, $(\varepsilon, 1-\varepsilon)$ of Θ_ε, there is a weight $w \in \mathbb{R}_+^{\mathcal{S}}$, for the same offer O, such that each open cell of O has mass at least ε, $\mathbf{s}_{ab}$ equals the vertex, and $\mathbf{s}_{bc} = \mathbf{s}_{ca} = (\frac{1}{2}, \frac{1}{2})$.*

Proof. The table gives the masses of the types and how they are shared among the parties. "r_c: ε each" means that each of the j_c far points on r_c receives the mass ε of that type; likewise for $[ab|c]$. The rest of the type goes to the party named.

type	mass	allocation
Vertex $(\varepsilon, \varepsilon)$		
abc, acb	$(1-t)\varepsilon$, $t\varepsilon$	a
bac, bca	$(1-t)\varepsilon$, $t\varepsilon$	b
$[c\|ab]$	$\frac{1}{2} - t\varepsilon$	r_c: ε each; rest to c
$[ab\|c]$	$\frac{1}{2} - 2\varepsilon + t\varepsilon$	$[ab\|c]$: ε each but the last, which takes the rest; if $j' = 0$, no party
Vertex $(1-\varepsilon, \varepsilon)$		
abc	$\frac{1}{2} - \varepsilon$	$[ab\|c]$: ε each; rest to a
acb	$t\varepsilon$	a
bac, bca	$(1-t)\varepsilon$, $t\varepsilon$	b
cab	$\frac{1}{2} - t\varepsilon$	r_c: ε each; rest to c
Vertex $(\varepsilon, 1-\varepsilon)$		
abc, acb	$(1-t)\varepsilon$, $t\varepsilon$	a
bac	$\frac{1}{2} - \varepsilon$	$[ab\|c]$: ε each; rest to b
bca	$t\varepsilon$	b
cba	$\frac{1}{2} - t\varepsilon$	r_c: ε each; rest to c

Masses and cells. All masses are non-negative: $t \in [\frac{1}{2}, 1]$, $\frac{1}{2} - \varepsilon \ge 0$, and the others by (C1) and (C2). At the vertex $(\varepsilon, \varepsilon)$, the cells of a and b have mass ε; the cell of c has mass $\frac{1}{2} - t\varepsilon - j_c\varepsilon \ge \varepsilon$ by (C1); the last point on $[ab|c]$ receives $\frac{1}{2} - 2\varepsilon + t\varepsilon - (j'-1)\varepsilon \ge \varepsilon$ by (C2). At the vertex $(1-\varepsilon, \varepsilon)$, the cell of a has mass $\frac{1}{2} - \varepsilon - j'\varepsilon + t\varepsilon \ge \varepsilon$ by (C2); the cell of b has mass ε; the cell of c is as before. The third vertex is the second with a and b exchanged. The far points receive at least ε. The total mass is 1 in each case; for instance $2\varepsilon + \frac{1}{2} - t\varepsilon + \frac{1}{2} - 2\varepsilon + t\varepsilon = 1$.

Shares at $(\varepsilon,\varepsilon)$. $s_{ab}=m(abc)+m(acb)=\varepsilon$ and $s_{ba}=m(bac)+m(bca)=\varepsilon$. $s_{bc}=m(abc)+m(bac)+m(bca)+m([ab|c])=(1-t)\varepsilon+(1-t)\varepsilon+t\varepsilon+\frac{1}{2}-2\varepsilon+t\varepsilon=\frac{1}{2}$. $s_{cb}=m(acb)+m([c|ab])=\frac{1}{2}$. $s_{ca}=m(bca)+m([c|ab])=\frac{1}{2}$. $s_{ac}=m(abc)+m(acb)+m(bac)+m([ab|c])=\frac{1}{2}$.

Shares at $(1-\varepsilon,\varepsilon)$. $s_{ab}=m(abc)+m(acb)+m(cab)=\frac{1}{2}-\varepsilon+t\varepsilon+\frac{1}{2}-t\varepsilon=1-\varepsilon$ and $s_{ba}=\varepsilon$. $s_{bc}=m(abc)+m(bac)+m(bca)=\frac{1}{2}$; $s_{cb}=m(acb)+m(cab)=\frac{1}{2}$; $s_{ca}=m(bca)+m(cab)=\frac{1}{2}$; $s_{ac}=m(abc)+m(acb)+m(bac)=\frac{1}{2}$.

Shares at $(\varepsilon,1-\varepsilon)$. $s_{ab}=\varepsilon$; $s_{ba}=\frac{1}{2}-\varepsilon+t\varepsilon+\frac{1}{2}-t\varepsilon=1-\varepsilon$; $s_{bc}=(1-t)\varepsilon+\frac{1}{2}-\varepsilon+t\varepsilon=\frac{1}{2}$; $s_{cb}=t\varepsilon+\frac{1}{2}-t\varepsilon=\frac{1}{2}$; $s_{ca}=\frac{1}{2}$; $s_{ac}=(1-t)\varepsilon+t\varepsilon+\frac{1}{2}-\varepsilon=\frac{1}{2}$.

Each type is hosted by a party that can host it, by the table of supports. □

Theorem S3 (Minimum size with ties)**.** *Let $k\geq 3$, $0<\varepsilon\leq 1/k$ and $\mathcal{M}\supseteq\mathcal{M}_1$. On $\mathcal{D}^{\mathrm{tie}}_{k,\varepsilon}(\mathcal{M})$, no rule satisfies anonymity, independence, non-imposition and non-nullness. Moreover $D_{ab}=\Theta_\varepsilon$ for every pair.*

Proof. A cross for the pair $\{a,b\}$. Fix O and $\mathcal{S}$ as above. The set P of weights $w\geq 0$ of sum 1 such that every open cell of O has mass at least ε and $\mathbf{s}_{bc}=\mathbf{s}_{ca}=(\frac{1}{2},\frac{1}{2})$ is a convex polyhedron, since its constraints are linear. Its image under the linear map $w\mapsto\mathbf{s}_{ab}$ is convex and contains the three vertices of Θ_ε (Lemma S6). So it contains Θ_ε. Hence for every $v\in\Theta_\varepsilon$ there is a distribution of $\mathcal{M}_1$ such that (x,O) lies in the domain, with $\mathbf{s}_{ab}=v$ and $\mathbf{s}_{bc}=\mathbf{s}_{ca}=(\frac{1}{2},\frac{1}{2})$. With Lemma S3, $D_{ab}=\Theta_\varepsilon$, and the cross of coordinate ab passes through $h=((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$, which is produced, with $v=(\frac{1}{2},\frac{1}{2})$.

The two other coordinates. The same constructions, with the names rotated ($a\to b\to c\to a$, hence far points on r_a and $[bc|a]$, then on r_b and $[ca|b]$), give the crosses of the coordinates bc and ca, through the same h. They use other offers, all containing T, which is allowed.

Conclusion. Lemma S1 makes the pair $\{a,b\}$ indifferent everywhere. The pair was arbitrary, so the rule is null. □

Take $k=4$ and $\varepsilon=\frac{1}{4}$: $t=1$, $j_c=0$, $j'=1$. At the vertex $(\frac{1}{4},\frac{1}{4})$, the distribution puts $\frac{1}{4}$ on the sector acb (party a), $\frac{1}{4}$ on bca (party b), $\frac{1}{4}$ on a segment of r_c (party c), and $\frac{1}{4}$ on a segment of $[ab|c]$ around the far point d (party d). Each cell has mass $\frac{1}{4}$. The shares are $\mathbf{s}_{ab}=(\frac{1}{4},\frac{1}{4})$ and $\mathbf{s}_{bc}=\mathbf{s}_{ca}=(\frac{1}{2},\frac{1}{2})$; for instance b against c is preferred by bca and $[ab|c]$, of mass $\frac{1}{2}$, and c against b by acb and $[c|ab]$, of mass $\frac{1}{2}$.

The mechanism, step by step. (1) A very indifferent value of the duel (a,b), such as $(\varepsilon,\varepsilon)$, needs almost every citizen to be indifferent between a and b. (2) These citizens need not be members of c: some of them are tied at the top between a and b (type $[ab|c]$), and they are members of a far party or of no party. (3) The duels (b,c) and (c,a) can therefore stay exactly tied, half and half. (4) The cross passes through the centre, and Lemma S1 concludes. Conditions (C1) and (C2) say that there is room for the far parties, and they amount exactly to $k\varepsilon\leq 1$.

Remark S3 (Other conventions for boundaries). (1) For $k\geq 4$, $j'\geq 1$, and the construction puts no mass on the boundaries of O: the polyhedron with the extra constraint "zero weight on the boundary segment" is still convex and contains the three vertices. The theorem therefore also holds if offers are required to give zero mass to boundaries.

(2) For $k=3$ with zero-mass boundaries, every citizen has a strict first choice in $O=T$. On a non-aligned triangle, the produced share vectors are exactly $\bigcup_{m\in M_\varepsilon}\tilde{B}(m)$: members of t lie in the two sectors of t or on the inner half-line r_t, which lies in the open cell of t, and every such family of masses is produced by $\mathcal{M}_1$. If $\varepsilon\geq\frac{1}{4}$, Theorem S2 applies, provided the global set D_{ab}, computed over all three-point offers with zero-mass boundaries, aligned offers included, equals $\Theta_\varepsilon=\bigcup_{m\in M_\varepsilon}X(m)$. It does. In every such offer, $s_{ab}\geq m_a\geq\varepsilon$, $s_{ba}\geq m_b\geq\varepsilon$ and $s_{ab}+s_{ba}\leq 1$, so $D_{ab}\subset\Theta_\varepsilon$; the reverse inclusion comes from a non-aligned triangle. Each non-aligned triangle is then a star triple: by Theorem S2, the products $\tilde{B}(m)$ have a common

point and cover the projections, which are the global sets D_{ab}, D_{bc}, D_{ca}. Every pair lies in a non-aligned triangle, and Theorem 1 (Remark S1) gives the impossibility. If $\varepsilon < \frac{1}{4}$, the same theorem gives a valid, non-imposed, non-null labelling on every non-aligned triangle, so no non-aligned triangle is a star triple. Whether a rule exists on the whole domain is open. A numerical observation, not a proof: on the abstract problem where five platforms form ten offered triples, each with all the shares $\bigcup_{m\in M_\varepsilon} \tilde{B}(m)$, a solver finds no rule on grids of step $1/10$ for $\varepsilon = 0.1$ and 0.2, while it finds one with four platforms. This suggests an impossibility through several triples.

Parties at the mean of their members, with ties

A citizen equidistant from several nearest platforms lies on a *boundary*. Three conventions are natural.

(a) *Member of no party.* The members of q are the citizens of the open cell $V_q(O)$. An offer is an equilibrium if each V_q has positive mass and mean q.

(b) *Equal sharing.* A citizen equidistant from r nearest platforms counts for $1/r$ in each of them. An offer is an equilibrium if each platform is the mean of its cell so weighted.

(c) *Zero-mass boundaries.* Only offers whose boundaries have zero mass are admitted; the fixed-point condition is then the same under every convention.

The fixed-point condition depends on the convention: the same (μ, O) may be an equilibrium under (a) and not under (b), since citizens on boundaries move the mean under (b) and not under (a). Convention (a) uses the notion of member used everywhere else, a citizen who strictly prefers her party to every other; it is the convention of Definition S1; and its equilibria contain those of (c). Convention (c) matters because the optimal selection always leads to it (Lemma S9).

Domains. Let $\mathcal{M} \supseteq \mathcal{M}_1$ be a set of distributions with a finite first moment (a finite second moment for the optimal selection). For $k \geq 3$, $\mathcal{D}_k^{(a)}(\mathcal{M})$ is the set of (x, O) with $\mu_x \in \mathcal{M}$, O a set of k distinct points, and O an equilibrium under (a). The domains $\mathcal{D}_k^{(b)}$ and $\mathcal{D}_k^{(c)}$ are defined in the same way; under (b) and (c) I also require that each open cell has positive mass, which rules out degenerate equilibria under (b) where a platform has only boundary citizens as "members". In all these domains, every equilibrium offer of a distribution is admitted.

Lemma S7 (Shares to cover)**.** *In $\mathcal{D}_k^{(a)}$, $\mathcal{D}_k^{(b)}$, $\mathcal{D}_k^{(c)}$, and for every selection among their equilibria, $D_{ab} \subset \Theta^\circ$.*

Proof. In the three domains the open cell of a has positive mass by definition, and its citizens strictly prefer a to b. Likewise for b. □

Let $T = \{a, b, c\}$ be an *equilateral* triangle, with centre ω. Each vertex t lies on the line (ωt), which is the bisector of the two other vertices, and the half-line r_t passes through t. A small cluster centred at t can therefore have members on both sides of (ωt), and on (ωt), without moving its mean.

Balanced cluster. For $t \in T$, masses $\alpha, \beta > 0$ and $\gamma \geq 0$, and small $\rho > 0$, the *cluster* $A_t(\alpha, \beta, \gamma; \rho)$ consists of a disc of mass α on one side of (ωt) and a disc of mass β on the other side, placed as in the proof of Lemma S25 so that their mean is t, and of a segment of (ωt) centred at t, of length at most 2ρ and mass γ. It lies in $\bar{B}(t, \rho)$ and has mean t. For ρ small, its two discs lie in the two sectors where t is first, and its segment lies on r_t.

Lemma S8 (Richness with ties, convention (a)). *Let $k \geq 3$ and $(p,q) \in \Theta^\circ$. There are $\mu \in \mathcal{M}_1$ and an offer $O \supset T$ of k points, an equilibrium of μ under (a), with*

$$\mathbf{s}_{ab} = (p,q), \qquad \mathbf{s}_{bc} = \mathbf{s}_{ca} = (\tfrac{1}{2},\tfrac{1}{2}).$$

The same holds after rotating the names.

Proof. Masses of the types. Let $\delta = \frac{1}{2}\min(p,q)$ and

$$x_1 = x_5 = \tfrac{p-\delta}{2}, \quad x_3 = x_6 = \tfrac{q-\delta}{2}, \quad x_2 = x_4 = \delta, \quad g = g' = \tfrac{1-p-q}{2},$$

for the types *abc* (x_1), *acb* (x_2), *bac* (x_3), *bca* (x_4), *cab* (x_5), *cba* (x_6), $[c|ab]$ (g), $[ab|c]$ (g'). All these masses are non-negative, and $x_1, \ldots, x_6 > 0$. Their sum is $(p+q-2\delta)+2\delta+(1-p-q) = 1$.

Shares. $s_{ab} = x_1 + x_2 + x_5 = p$ and $s_{ba} = x_3 + x_4 + x_6 = q$. $s_{bc} = x_1 + x_3 + x_4 + g' = \frac{p+q-2\delta}{2} + \delta + \frac{1-p-q}{2} = \frac{1}{2}$; $s_{cb} = x_2 + x_5 + x_6 + g = \frac{1}{2}$; $s_{ca} = x_4 + x_5 + x_6 + g = \frac{1}{2}$; $s_{ac} = x_1 + x_2 + x_3 + g' = \frac{1}{2}$.

Placement, $k = 3$. Take the clusters $A_a(x_1, x_2, 0)$, $A_b(x_3, x_4, 0)$, $A_c(x_5, x_6, g)$, with the disc of mass x_1 on the side of the sector *abc*, and so on, and put the mass g' uniformly on a segment of the half-line $[ab|c]$. For ρ small each cluster lies in the open cell of its vertex; the segment on $[ab|c]$ lies on the boundary between a and b, and its citizens are members of no party. The restriction of μ to V_t is the cluster of t, of positive mass and mean t. This is an equilibrium under (a).

Placement, $k \geq 4$. Let u be the unit vector of r_c, and $d_j = \omega + \tau_j u$, $j = 1, \ldots, k-3$, with τ_j large, distinct and pairwise at least 1 apart, and far clusters of radius less than $\frac{1}{4}$, as Lemma S4(ii) requires. Let $\theta = \frac{1}{k-1}$. The cluster of c becomes $A_c((1-(k-3)\theta)(x_5, x_6, g))$, and each d_j receives a copy $\theta(x_5, x_6, g)$: two balanced discs on either side of L_{ab} and a segment of L_{ab} centred at d_j. By Lemma S4 with $j' = 0$, for τ_j large these citizens have types *cab*, *cba*, $[c|ab]$; the clusters of a, b, c and the segment on $[ab|c]$, taken near ω, keep their place relative to O. The types, hence the shares, are unchanged, and each open cell is a cluster whose mean is its platform. □

Theorem S4 (Convention (a), all equilibria). *For every $k \geq 3$, on $\mathcal{D}_k^{(a)}(\mathcal{M})$ with $\mathcal{M} \supseteq \mathcal{M}_1$, no rule satisfies anonymity, independence, non-imposition and non-nullness.*

Proof. Let $\{a, b\}$ be a pair of points of the plane, and c such that $T = \{a, b, c\}$ is equilateral. By Lemma S8, the pair is offered, $h = ((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})) \in V_T$, and $(w, h_2, h_3) \in V_T$ for every $w \in \Theta^\circ$. The rotation by $2\pi/3$ about ω maps T to itself ($a \to b \to c \to a$) and preserves $\mathcal{M}_1$ and the equilibrium condition; applied to the constructions, it gives the crosses of the coordinates *bc* and *ca*. By Lemma S7, $D_{ab}, D_{bc}, D_{ca} \subset \Theta^\circ$. Lemma S1 makes the pair indifferent everywhere. Every pair of points is a side of an equilateral triangle, so the rule is null. □

The mechanism is the one of Theorem S3. The citizens tied at the top between a and b count in no mean; they absorb the indifference of the duel (a, b) without making c larger. The fixed-point condition does not interfere, because the equilateral triangle lets each cluster split between its two sectors around its platform.

Proposition S1 (Convention (b), all equilibria). *Theorem S4 also holds on $\mathcal{D}_k^{(b)}(\mathcal{M})$, for every $k \geq 3$.*

Proof. Take the masses of Lemma S8. Write $a = \omega + r(e_1 + e_2)$, where e_1 and e_2 are the unit vectors of the half-lines $[ab|c]$ and $[ca|b]$, which bound the cell of a, an angle of 120°; so $|e_1 + e_2| = 1$ and $r = |\omega a|$. Put the mass g' on a small segment of $[ab|c]$ centred at $y = \omega + s e_1$, with $0 < s \leq r$. Under (b) this segment counts for $g'/2$ in the cell of a and for $g'/2$ in that of

b. The weighted cell of a has mean a if and only if the members of a, of mass $m_a = x_1 + x_2$, have mean

$$c_a = a + \kappa(a - y), \qquad \kappa = \frac{g'}{2m_a}.$$

Now $c_a = y + (1+\kappa)(a-y) = \omega + \big(s + (1+\kappa)(r-s)\big)e_1 + (1+\kappa)r\, e_2$. Both coefficients are positive, so c_a lies in the open cell of a. Since the two masses x_1, x_2 are positive, one can place two small discs, one in each sector of a, possibly far, with mean c_a: the set of possible means $\frac{x_1P_1 + x_2P_2}{m_a}$, with P_1 and P_2 in the two open sectors, is ω plus the sum of two open cones, that is, the open cell of a (as in Lemma 7(b)). The same holds for b. The cell of c does not touch $[ab|c]$; its cluster is unchanged. The types, hence the shares, are those of Lemma S8. For $k \geq 4$, the far points are added as in that lemma, far enough that the discs of a and b, which may be far from ω, stay in their cells. The conclusion follows as in Theorem S4, with Lemma S7. □

For a distribution μ with a finite second moment, $\mathrm{Opt}(\mu)$ is the set of sets Q of at most k points that minimize $\Delta_\mu(Q) = \int \min_{q \in Q} |z - q|^2 \, d\mu(z)$. An optimal selection picks $\sigma(\mu) \in \mathrm{Opt}(\mu)$ when this set is non-empty and the support of μ has at least k points; the latter holds for every distribution of $\mathcal{M}_1$.

Lemma S9 (Optimal offers do not charge their boundaries)**.** *Let μ have a finite second moment and a support with at least k points, and let $Q \in \mathrm{Opt}(\mu)$. Then $|Q| = k$; the union of the boundaries of the cells of Q has mass zero; and each open cell has positive mass and has its platform as mean. In particular Q is an equilibrium under (a), (b) and (c).*

This is a classical result of quantization theory; see Graf and Luschgy (2000, *Foundations of Quantization for Probability Distributions*, Lecture Notes in Mathematics 1730, Springer, Section 4). I give a complete proof.

Proof. Assignments. An assignment is any measurable map ϕ that sends each citizen to one of her nearest platforms. For every assignment, $\Delta_\mu(Q) = \sum_q \int_{\phi^{-1}(q)} |z - q|^2 \, d\mu$. If $\mu(\phi^{-1}(q)) > 0$, then q is the mean of $\phi^{-1}(q)$. Otherwise, replacing q by that mean while keeping the assignment lowers the sum strictly, and reassigning each citizen to her nearest point does not raise it; this contradicts optimality.

Boundaries. Suppose the union of the boundaries has positive mass. There are then $q \neq q'$ in Q such that $E = \{z : \ q \text{ and } q' \text{ are among the nearest points of } Q\}$ has positive mass. E is closed and lies in the bisector L of q and q'. Take two assignments that differ only on E: the first sends E to q, the second to q'. Let G be the set sent to q by the second. By the first, q is the mean of $G \cup E$. If $\mu(G) > 0$, q is also the mean of G, by the second, so the mean of E is q; if $\mu(G) = 0$, the mean of E is q directly. But the mean of a measure carried by L lies on L, and $q \notin L$: a contradiction.

Cells and number of points. Suppose $|Q| < k$, or some open cell V_q has mass zero. In the second case, removing q does not change the distortion: the points of the boundary of the cell of q have another nearest platform at the same distance. Either way there is a set Q' of at most $k - 1$ points with $\Delta_\mu(Q') = \Delta_\mu(Q)$. Since the support has at least k points, it contains some $y \notin Q'$. On a small ball around y, of positive mass, y is strictly closer than Q', so $\Delta_\mu(Q' \cup \{y\}) < \Delta_\mu(Q)$: a contradiction. The means come from the first step, since with zero-mass boundaries all assignments agree almost everywhere. □

Under the optimal selection, the convention on boundaries is therefore irrelevant. But the citizens tied at the top relative to O disappear. For $k \geq 4$, the only ones left are citizens tied at the top relative to a *triple* $T \subset O$, who are members of another party.

Theorem S5 (Optimal offer and zero-mass boundaries, $k \geq 4$)**.** *Let $k \geq 4$ and $\mathcal{M} \supseteq \mathcal{M}_1$, with a finite second moment. Under any optimal selection, and also on $\mathcal{D}_k^{(c)}(\mathcal{M})$ (all equilibria with zero-mass boundaries), no rule satisfies anonymity, independence, non-imposition and non-nullness.*

Proof. Construction. Let $(p, q) \in \Theta^\circ$, T equilateral, and take the masses of Lemma S8. Let d_1 be a point of the open half-line $[ab|c]$, and $\xi = \frac{1}{2}\min(x_1, x_3) > 0$. The cluster of d_1 consists of two discs of mass ξ, symmetric with respect to L_{ab}, one in abc and the other in bac, and of a segment of L_{ab} centred at d_1, of mass g'; its mean is d_1. The argument of Lemma S4(iii) places the discs in these sectors: it holds for every point of the open half-line, since the distance from d_1 to L_{ac} and L_{bc} is positive, and for a small enough ball. The clusters of a and b become $A_a(x_1 - \xi, x_2, 0)$ and $A_b(x_3 - \xi, x_4, 0)$. The $k - 4$ other points are placed on r_c with copies $\theta(x_5, x_6, g)$ of the cluster of c, as in Lemma S8, with $\theta = \frac{1}{k-1}$ and only $k - 4$ copies, so that the cluster of c keeps the fraction $1 - (k - 4)\theta > 0$. These points on r_c are pairwise at least 1 apart, their clusters have radius less than $\frac{1}{4}$, and Lemma S4(iii) guarantees their type relative to T for τ large. The types are unchanged, and the shares are still (p, q), $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$. No citizen lies on a boundary.

Selection. The k platforms are distinct. Let L be their smallest mutual distance and $\eta > 0$ the smallest cluster mass; neither depends on ρ. For $\rho < \sqrt{\eta}L/(3(1+\sqrt{\eta}))$, Lemma S23 shows that O is the unique element of $\mathrm{Opt}(\mu)$ and that its cells are the clusters. The selection therefore picks O. It is also an equilibrium with zero-mass boundaries, hence a pair of $\mathcal{D}_k^{(c)}$.

Conclusion. By Lemmas S9 and S7, $D_{ab} \subset \Theta^\circ$, and the construction produces the whole cross of the coordinate ab, and h. Rotations give the two other coordinates. Lemma S1 concludes as in Theorem S4. □

Take $k = 4$ and $(p, q) = (0.2, 0.2)$: $\delta = 0.1$, $x_1 = x_3 = x_5 = x_6 = 0.05$, $x_2 = x_4 = 0.1$, $g = g' = 0.3$, $\xi = 0.025$. The cluster of d_1, on $[ab|c]$, has mass 0.35: 0.3 of citizens tied at the top between a and b, and two discs of 0.025. These 0.3 citizens are strictly closer to d_1 than to a, b and c; they are members of d_1, which every convention allows, and the optimal offer is $\{a, b, c, d_1\}$.

For $k = 3$, under (c) or under the optimal selection, $O = T$, and almost every citizen has a strict first choice in T. The shares produced on T thus lie in $\bigcup_m \tilde{B}(m)$, m in the open simplex: this is free arbitration with ties, with $\bar{m}_i = 1$, and Theorem S2 shows that on an isolated triple the star lemma cannot serve. The next result says that this remains true with the shares produced in all offers.

Proposition S2 (No star triple)**.** *Consider a domain where every offer has three points, gives zero mass to its boundaries and positive mass to its open cells. Let $\{a, b\}$ be a pair such that D_{ab} contains values (η, η) for arbitrarily small $\eta > 0$. Then, for every triple $T \supset \{a, b\}$ and every $h \in V_T$, the cross of coordinate ab at h is not contained in V_T. In particular no triple containing $\{a, b\}$ is a star triple. This is the case of the optimal selection and of $\mathcal{D}_3^{(c)}$.*

Proof. Suppose $((\eta, \eta), h_2, h_3) \in V_T$. At the pair that produces it, the members of a prefer a to b and those of b prefer b to a: $m_a, m_b \leq \eta$, hence $m_c \geq 1 - 2\eta$, since boundaries have zero mass. The members of c prefer c to b, so the second coordinate of h_2 is at least $1 - 2\eta$. If this holds for arbitrarily small η, then $h_2 = (0, 1)$. But $h \in V_T$, and at the pair that produces h, the members of b, of positive mass, prefer b to c: the first coordinate of h_2 is positive. A contradiction. By Remark S2, a star triple would have such crosses.

For the optimal selection and for $\mathcal{D}_3^{(c)}$, take clusters around an equilateral triangle, with a and b of mass η and c carried by a segment of r_c centred at c, of mass $1 - 2\eta$. They produce (η, η), and Lemma S23 makes T the unique optimal set. □

Remark S4 (Three parties with zero-mass boundaries: open). Whether a rule exists for parties at the mean of their members with $k = 3$ and zero-mass boundaries, under convention (c) or under the optimal selection, is open. The star method fails on every triple (Proposition S2), and an impossibility would need an argument that combines several triples sharing pairs. Numerical checks on grids, which are not proofs, find no obstruction. On the abstract problem where each triple produces all the shares $\bigcup_m \tilde{B}(m)$, a rule exists with four platforms and none with five. But that richness is not produced by the mean-of-members model: under (c), if z is strictly closer to x than to y, some member of z prefers x to y, since otherwise the members of z, hence their mean, lie in the closed half-plane on the side of y. With the exact produced sets, a solver finds a rule for the regular pentagon, for random sets of five to nine points, and for pieces of the triangular lattice with six, seven and ten points.

S1.2 Collinear triples and the mixed case

Lemma S10. *For an offer S, the following are equivalent: (a) S is not aligned; (b) every pair of S belongs to a non-aligned triple of S. If S is aligned, all its triples are aligned.*

Proof. (a)⇒(b): for $a \neq b$, $S \not\subset (ab)$, so some $c \in S$ lies off (ab). (b)⇒(a) and the last sentence are clear. □

For a fixed offer there is thus no intermediate case: "every triple containing a given pair is aligned" means "the whole offer lies on a line".

Proposition S3 (The median rule on a line)**.** *Let S be an offer aligned on an oriented line ℓ. For a distribution μ, let t be the coordinate of the orthogonal projection on ℓ, $G(u) = \mu(t \leq u)$, and $c^* = \min\{u : G(u) \geq \frac{1}{2}\}$, the lower median; the minimum exists since G is right-continuous. For a before b on ℓ, let*

$$a \succ b \iff s_{ba} \leq \tfrac{1}{2}, \qquad b \succ a \text{ otherwise.}$$

This rule is anonymous and independent, and it ranks S by distance to c^*, ties going to the leftmost point; hence it ranks S by a strict total order. On $\mathcal{M}_0$, as on the set of all distributions, it is non-imposed and non-null.

Proof. $|z-a|^2 - |z-b|^2$ is a strictly increasing affine function of $t(z)$ that vanishes at the midpoint m_{ab} of $[a,b]$. So $H_{ba} = \{t > m_{ab}\}$ and $s_{ba} = 1 - G(m_{ab})$. Hence $a \succ b$ if and only if $G(m_{ab}) \geq \frac{1}{2}$, if and only if $m_{ab} \geq c^*$, if and only if $|c^* - a| \leq |c^* - b|$; the last step is checked directly according to the position of c^* relative to $a < b$. The relation is thus the lexicographic order by distance to c^*, then by position, which is a strict total order. It depends only on shares, which gives anonymity and independence. Non-imposition: a distribution concentrated near a, or near b, and charging no line, gives $a \succ b$, or $b \succ a$. □

Corollary S2 (Fixed offer: the exact criterion)**.** *Let S be a fixed offer with $|S| \geq 3$, and $\mathcal{M} = \mathcal{M}_0$ or $\mathcal{M} \supseteq \mathcal{M}_1$. A rule satisfying anonymity, independence, non-imposition and non-nullness exists if and only if S is aligned.*

Proof. Theorem S1 and Proposition S3. □

The median rule is not neutral: it uses an orientation of the line to break ties of distance. Simple majority ($a \succ b$ if $s_{ab} > s_{ba}$, and $a \sim b$ otherwise) does not work in general. With half of the citizens near 0 and half near 2.3, and $S = \{0, 2, 4\}$ on a line, it gives $0 \sim 2$, $2 \succ 4$ and $0 \sim 4$, which is not transitive.

Proposition S4 (All offers aligned)**.** *Suppose every offer of the domain is aligned, the line possibly changing from one offer to another. Choose an orientation of each line of the plane, and apply to each pair $\{a, b\}$ the rule of Proposition S3 relative to the line (ab). The resulting rule is anonymous, independent across offers, and gives a strict total order on every offer. It is non-null as soon as a pair is offered. It is non-imposed if and only if, for each offered pair with a before b, some pair (x, O) of the domain has $s_{ba} \leq \frac{1}{2}$ and another has $s_{ba} > \frac{1}{2}$.*

Proof. The label of $\{a, b\}$ depends only on $\mathbf{s}_{ab}$ and on the line (ab), which is determined by the pair: this gives anonymity and independence. On an offer aligned on ℓ, all pairs use ℓ and its orientation, so the rule is the order of Proposition S3. The last sentence translates non-imposition for a strict rule. □

When no line is charged, $s_{ba} \leq \frac{1}{2}$ is $s_{ab} \geq \frac{1}{2}$. With the orientation given by the lexicographic order $<_{\text{lex}}$ of the plane, the rule then reads: $p \succ q$ if $s_{pq} > \frac{1}{2}$, or if $s_{pq} = \frac{1}{2}$ and $p <_{\text{lex}} q$.

The mixed case. When some pairs are offered only in aligned offers, Theorem 1 says nothing about them. A pair is *null* if $\ell_{ab} \equiv \mathsf{I}$ on D_{ab}; a null pair satisfies non-imposition. Let $\mathcal{N}$ be the set of offered pairs that belong to at least one star triple. The mixed case is the case where $\mathcal{N}$ does not contain every offered pair. Shares may be numbers or vectors: nothing below depends on it. By Theorem 1 and Remark S1, every pair of $\mathcal{N}$ is null under anonymity, independence and non-imposition; more precisely, if T is a star triple, non-imposition of the three pairs of T makes all three null.

Corollary S3 (The exact form of the impossibility)**.** *(a) No rule satisfies anonymity, independence, non-imposition, and is non-null on a pair of $\mathcal{N}$.*

(b) If $\mathcal{N}$ contains every offered pair, then no rule satisfies anonymity, independence, non-imposition and non-nullness.

(c) Call strict non-imposition *the requirement that each direction of each offered pair wins strictly at least once: for each pair, F and S are both taken on D_{ab}. It implies non-imposition. If $\mathcal{N} \neq \emptyset$, then no rule satisfies anonymity, independence and strict non-imposition. The same holds for strong robust non-imposition (Axiom 5).*

Proof. (a) and (b) follow from Theorem 1. (c) Strict non-imposition excludes $\ell \equiv \mathsf{F}$ and $\ell \equiv \mathsf{S}$, so it implies non-imposition. Theorem 1 makes the pairs of $\mathcal{N}$ null, which contradicts strict non-imposition on them. Strong robust non-imposition implies strict non-imposition. □

With strict or strong robust non-imposition, a single star triple is enough, and the mixed case raises no problem. The question exists only with the weak versions, Wilson's or weak robust non-imposition, which let a null pair pass. For instance, take a fixed non-aligned offer $\{a, d, e\}$ with $\mathcal{M} = \mathcal{M}_0$, and suppose that every offer of the domain containing a pair of $\{a, d, e\}$ comes with distributions that charge no line. The triple is then a star triple, by the cross at h of Theorem S1, and if $d \succ e$ is required to happen at least once, no rule exists, whatever the other offers. The proviso matters: if another offer containing de (or ad, ae) has indifferent citizens, D_{de} leaves the hypotenuse $p + q = 1$, the crosses of $\mathcal{M}_0$ no longer cover it, and the triple need not be a star triple.

Example S3 (A common party is not enough)**.** Let a, b, c be three distinct points on a line L, and d, e two points such that a, d, e are not aligned and $d, e \notin \{b, c\}$. The domain is

$$\mathcal{D} = \{(x, \{a, b, c\}) : \mu_x \in \mathcal{M}_0\} \cup \{(x, \{a, d, e\}) : \mu_x \in \mathcal{M}_0\}.$$

Then $\mathcal{N} = \{ad, ae, de\}$, and the following rule satisfies anonymity, independence, validity, non-imposition and non-nullness: on $\{a, b, c\}$, the median rule of Proposition S3 relative to L; on $\{a, d, e\}$, total indifference.

Proof. The triple $\{a, d, e\}$ is a star triple: the pair ad is offered only in $\{a, d, e\}$, so $D_{ad} = [0, 1]$ is computed on that offer alone, and Lemma S2 gives the cross; likewise for ae and de. No pair is offered in both offers, which share only a. The labels are therefore well defined pair by pair: median for ab, bc, ca, and I for ad, ae, de. They depend only on the shares of the pair, which gives anonymity and independence. Each offer receives a weak order: the median order, or indifference. The median rule is non-imposed on $\mathcal{M}_0$; null pairs satisfy non-imposition. The rule is strict on $\{a, b, c\}$. □

The mechanism, step by step. (1) The star triple makes its three pairs null, and only them. (2) An offer that contains none of these pairs is untouched, even if it shares a party. (3) On an aligned offer, the median rule gives a strict order, and non-nullness holds there. Offers of two points are aligned, and give an even simpler case: if the pair $\{b, c\}$ is offered only alone, majority on that pair is a non-null rule, the other pairs being null, provided D_{bc} contains shares on both sides of $\frac{1}{2}$, or the share $\frac{1}{2}$, so that majority is non-imposed.

Lemma S11 (Transmission)**.** *Let $T = \{a, b, c\}$ be an offered triple, with cyclic coordinates $v = (v_1, v_2, v_3) = (\mathbf{s}_{ab}, \mathbf{s}_{bc}, \mathbf{s}_{ca})$. Suppose $\ell_{ab} \equiv \mathsf{I}$ on $\pi_1(V_T)$ and the rule valid on V_T. Write $\bar{\lambda}$ for the label λ with F and S exchanged, $\bar{\mathsf{I}} = \mathsf{I}$.*

(i) At every $v \in V_T$, $\ell_{ca}(v_3) = \overline{\ell_{bc}(v_2)}$. In words: $a \sim b$, and c is first, last, or tied with both.

(ii) Let G_T be the bipartite graph whose vertices are the values in $\pi_2(V_T)$ and those in $\pi_3(V_T)$, a value v_2 being linked to a value v_3 if $(v_1, v_2, v_3) \in V_T$ for some v_1. On each connected component of G_T, ℓ_{bc} equals a constant λ and ℓ_{ca} equals $\bar{\lambda}$.

(iii) If G_T is connected, $\pi_2(V_T) = D_{bc}$, and ℓ_{bc} is non-imposed, then $\ell_{bc} \equiv \mathsf{I}$ on D_{bc} and $\ell_{ca} \equiv \mathsf{I}$ on $\pi_3(V_T)$. If moreover $\pi_3(V_T) = D_{ca}$, both pairs are null.

Proof. (i) At $v \in V_T$ the labels are $(\mathsf{I}, \ell_{bc}(v_2), \ell_{ca}(v_3))$. A valid triple containing an I is $(\mathsf{I}, \mathsf{I}, \mathsf{I})$ or contains an F and an S; with one I in front, the last two are (I, I), (F, S) or (S, F). This is $\ell_{ca} = \overline{\ell_{bc}}$. The triple $(\mathsf{I}, \mathsf{F}, \mathsf{S})$ says $b \succ c$ and $a \succ c$: c is last; $(\mathsf{I}, \mathsf{S}, \mathsf{F})$ puts c first.

(ii) By (i), two linked vertices carry labels that are bars of each other. Along a path $v_2 \to v_3 \to v_2'$, $\ell_{bc}(v_2') = \overline{\overline{\ell_{bc}(v_2)}} = \ell_{bc}(v_2)$.

(iii) By (ii), ℓ_{bc} is constant on $\pi_2(V_T) = D_{bc}$. Non-imposition excludes F and S, so the constant is I. Then $\ell_{ca} = \bar{\mathsf{I}} = \mathsf{I}$ on $\pi_3(V_T)$. □

Example S4 (A common pair makes an aligned offer null)**.** Let a, b, c have abscissae $0, 1, 2$ on a line L, and let d lie off L. The domain is

$$\mathcal{D} = \{(x, \{a, b, c\}) : \mu_x \in \mathcal{M}_0\} \cup \{(x, \{a, b, d\}) : \mu_x \in \mathcal{M}_0\}.$$

The pairs bc and ca are offered only in the aligned offer $\{a, b, c\}$. Yet every rule satisfying anonymity, independence and non-imposition is null, and no rule satisfies the four axioms.

Proof. The triple $\{a, b, d\}$ is a star triple. The pair ab is offered in both offers, so $D_{ab} \subset [0, 1]$. On $\{a, b, d\}$, which is not aligned, Lemma S2 produces $(w, \frac{1}{2}, \frac{1}{2})$ for every $w \in [0, 1]$, and its rotations; these crosses pass through $h = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and cover D_{ab} and $D_{bd} = D_{da} = [0, 1]$. By Theorem 1, ab, bd and da are null.

The shares on the line. Let t be the abscissa of the projection on L and $G(u) = \mu(t \leq u)$, continuous since μ charges no line. The midpoints are $\frac{1}{2}$ for ab, $\frac{3}{2}$ for bc and 1 for ac. So

$s_{ab} = G(\frac{1}{2}) = p$, $s_{bc} = G(\frac{3}{2}) = r$ and $s_{ca} = 1 - G(1) = 1 - q$. Every sequence $0 \le p \le q \le r \le 1$ is produced in $\mathcal{M}_0$, with masses p, $q-p$, $r-q$, $1-r$ on small discs in the strips $t < \frac{1}{2}$, $\frac{1}{2} < t < 1$, $1 < t < \frac{3}{2}$, $t > \frac{3}{2}$. Hence

$$V_T = \{(p,\, r,\, 1-q) :\ 0 \le p \le q \le r \le 1\}, \qquad \pi_2(V_T) = \pi_3(V_T) = [0,1] = D_{bc} = D_{ca}.$$

The graph is connected. The value $v_2 = r$ is linked to $v_3 = y$ if and only if there is $q \le r$ with $y = 1 - q$, that is, $y \ge 1 - r$. The value $y = 1$ is linked to every value r, and every value y is linked to $r = 1$.

Conclusion. By Lemma S11(iii), bc and ca are null. The five offered pairs are null. □

The mechanism, step by step. (1) The star triple $\{a, b, d\}$ forces $a \sim b$ in every offer, on the line included. (2) On the line, with $a \sim b$, transitivity leaves three outcomes: c first, c last, or all tied. (3) Every value of s_{bc} coexists with the value $s_{ca} = 1$, and every value of s_{ca} with $s_{bc} = 1$. So the outcome cannot change from one point to another. (4) "c always first" and "c always last" are imposed. Only indifference everywhere is left.

Example S5 (A common pair, but restricted shares)**.** Keep a, b, c, d and L as in Example S4, and let

$$\begin{aligned}\mathcal{D} = \{(x, \{a,b,c\}) :\ \mu_x \in \mathcal{M}_0,\ G(1) > \tfrac{1}{2}\} \cup \{(x, \{a,b,d\}) :\ \mu_x \in \mathcal{M}_0\} \\ \cup \{(x, O) :\ \mu_x \in \mathcal{M}_0,\ O = \{b,c\} \text{ or } O = \{c,a\}\}.\end{aligned}$$

The rule "ab, bd, da null; $b \succ c$ if $s_{bc} > \frac{1}{2}$, $c \succ b$ if $s_{bc} < \frac{1}{2}$, $b \sim c$ otherwise; likewise for c, a with s_{ca}" satisfies anonymity, independence, validity, non-imposition and non-nullness.

Proof. $\{a, b, d\}$ is still a star triple, by the same proof, and its pairs are null, as they must be. On $\{a, b, c\}$, $s_{bc} = G(\frac{3}{2}) \ge G(1) > \frac{1}{2}$ gives $b \succ c$, and $s_{ca} = 1 - G(1) < \frac{1}{2}$ gives $a \succ c$; with $a \sim b$, this is the weak order $a \sim b \succ c$. On the two-point offers every label is a weak order. The two-point offers produce $s_{bc} \in [0, 1]$ and $s_{ca} \in [0, 1]$, so each direction wins strictly. The rule is strict on bc and ca. □

Here $\pi_2(V_T) = (\frac{1}{2}, 1] \ne D_{bc} = [0, 1]$, and the covering hypothesis of Lemma S11(iii) fails. On the aligned triple the rule imposes "c last"; non-imposition is met elsewhere, on the two-point offers. The non-imposition of a pair may be met outside the triple under study.

Definition S2 (Closure)**.** Let $\mathcal{N}_0 = \mathcal{N}$. An offered pair p belongs to $\mathcal{N}_{n+1}$ if it belongs to $\mathcal{N}_n$, or if there is an offered triple $T \ni p$, with pairs p, q, r, such that $q \in \mathcal{N}_n$, the linking graph of T between the values of p and those of r (defined as in Lemma S11, with q as first coordinate) is connected, and the values of p produced on T are all those of D_p. The *closure* is $\bar{\mathcal{N}} = \bigcup_n \mathcal{N}_n$.

Theorem S6 (Nullity on the closure)**.** *Let a rule satisfy anonymity, independence and non-imposition. Every pair of $\bar{\mathcal{N}}$ is null. In particular, if $\bar{\mathcal{N}}$ contains every offered pair, no rule satisfies anonymity, independence, non-imposition and non-nullness.*

Proof. By induction on n. For $n = 0$, this is Theorem 1. If the pairs of $\mathcal{N}_n$ are null and p enters $\mathcal{N}_{n+1}$ through T and $q \in \mathcal{N}_n$, Lemma S11(iii), with q as first coordinate after a cyclic renaming, makes p null; the bar is symmetric, so the order of p and r does not matter. A pair of $\bar{\mathcal{N}}$ lies in some $\mathcal{N}_n$. □

The closure is not always everything: in Example S3, $\bar{\mathcal{N}} = \mathcal{N}$; in Example S5, $\bar{\mathcal{N}} = \{ab, bd, da\}$; in Example S4, $\bar{\mathcal{N}}$ is the set of the five pairs. A partial converse covers Example S3.

Proposition S5 (Separate domains)**.** *Suppose every offer O of the domain is of one of two kinds: all pairs of O lie in $\mathcal{N}$; or O is aligned and no pair of O lies in $\mathcal{N}$. Take the rule "indifference on offers of the first kind; the rule of Proposition S4 on offers of the second kind". It is anonymous, independent and valid. It is non-imposed if and only if, for each offered pair $\{a, b\} \notin \mathcal{N}$, with a before b on the oriented line (ab), some pair of the domain has $s_{ba} \leq \frac{1}{2}$ and another has $s_{ba} > \frac{1}{2}$. It is non-null as soon as a pair outside $\mathcal{N}$ is offered.*

Proof. A pair of $\mathcal{N}$ appears only in offers of the first kind, and a pair outside $\mathcal{N}$ only in offers of the second kind. The label of a pair is therefore well defined: I in the first case, the median label relative to the line (ab) in the second. It depends only on the shares of the pair. Each offer receives a weak order: indifference, or the median order, which is a strict total order since the line is that of the offer. Null pairs satisfy non-imposition; for the others, the condition is the one stated. □

Between Proposition S5 and Theorem S6 lies a zone that I do not characterize: offers that mix pairs of $\bar{\mathcal{N}}$ and pairs outside it, without the transmission going all the way. Example S5 lies there: a rule exists, but it is not of the form of Proposition S5, since it imposes "c last" on the aligned offer.

These results settle the hypotheses of the impossibility theorems. (1) The sharpest statement, with no hypothesis on the other pairs, replaces "non-null" by "non-null on a pair that belongs to a star triple"; impossibility then always holds (Corollary S3(a)). (2) To keep plain non-nullness, every offered pair must lie in $\mathcal{N}$ (Theorem 1) or, more weakly, in $\bar{\mathcal{N}}$ (Theorem S6), and Example S3 shows that this cannot be dropped. (3) With strict non-imposition, $\mathcal{N} \neq \emptyset$ suffices (Corollary S3(c)). (4) "Non-null on a pair of an offered non-aligned triple" is false in general: a non-aligned triple need not be a star triple and may carry a rule (Proposition 5). It holds in models where every offered non-aligned triple is a star triple. This is the case of the mean-of-members model with all equilibria and of the minimum-size model with all configurations: for every offered pair and every c off the line, the triple is offered and is a star triple (Appendix B). Under the optimal selection, the proof uses a particular third point, which makes an equilateral triangle; it still gives that $\mathcal{N}$ contains every pair, but says nothing of the other triples. In all these models the mixed case does not arise. It arises when the set of offers is restricted, as in Remark S10, or under selections.

S1.3 Equilibrium selection

In this subsection distributions charge no line. A *selection* is a map σ, defined on a set $\mathrm{dom}(\sigma)$ of distributions, with $\sigma(\mu) \in \mathrm{Eq}(\mu)$. In the minimum-size model, $\mathrm{Eq}(\mu)$ is the set of sets of k distinct points whose cells all have mass at least ε, $0 < \varepsilon \leq 1/k$. The selected domain $\mathcal{D}_\sigma = \{(x, \sigma(\mu_x)) : \mu_x \in \mathrm{dom}(\sigma)\}$ satisfies Assumption 1, so Lemma 1 and Theorem 1 apply, with D_{ab} and V_T computed on $\mathcal{D}_\sigma$. Under a selection fewer share vectors are produced on a triple, but there are also fewer shares to cover.

In the minimum-size model, a selection *recognizes obvious clusters* if it satisfies Definition S4 for every $\eta \geq \varepsilon$; for $\eta < \varepsilon$ the cluster centres are not an equilibrium, and nothing is asked. The selection "optimal offer among the offers whose cells have mass at least ε" recognizes obvious clusters, with $\bar{\rho}(\eta, L) = \sqrt{\eta}L/(3(1 + \sqrt{\eta}))$. Indeed, by Lemma S23, the cluster centres Z form the unique unconstrained minimizer, and its cells are the clusters, of mass at least $\eta \geq \varepsilon$; so Z satisfies the constraint, and it is the unique constrained minimizer.

Perverse selections

Without the cluster property, a selection may pick offers where arbitration is never free. The first example lives in the minimum-size model with three parties, on distributions with a finite

second moment.

The aligned selection. For μ, let $L(\mu)$ be the line through the mean of μ directed by an eigenvector of the largest eigenvalue of the covariance; if the covariance is isotropic, proportional to the identity, let $L(\mu)$ be horizontal. The covariance has rank 2, since a covariance of rank 1 would mean that μ is carried by a line. A non-isotropic 2×2 covariance has two distinct eigenvalues, so the principal direction is well defined, up to sign. Let F be the distribution function of the projection of μ on $L(\mu)$, oriented; F is continuous, since μ charges no line orthogonal to $L(\mu)$. Let t_1 (resp. t_2) be the midpoint of the interval $\{F = 1/3\}$ (resp. $\{F = 2/3\}$), which is non-empty and compact; $t_1 < t_2$. On $L(\mu)$ let

$$p_1 = \tfrac{3t_1 - t_2}{2}, \qquad p_2 = \tfrac{t_1 + t_2}{2}, \qquad p_3 = \tfrac{3t_2 - t_1}{2},$$

and $\sigma(\mu) = \{p_1, p_2, p_3\}$. The midpoints of p_1p_2 and p_2p_3 are t_1 and t_2; the cells are the three strips cut at t_1 and t_2, of mass $\frac{1}{3}$ each. So $\sigma(\mu) \in \mathrm{Eq}(\mu)$ for every $\varepsilon \leq \frac{1}{3}$. The orientation of $L(\mu)$ does not matter: reversing it exchanges t_1 and t_2, hence p_1 and p_3, and leaves p_2 and $\sigma(\mu)$ unchanged.

Proposition S6 (A perverse selection)**.** *In the minimum-size model with $k = 3$ and $0 < \varepsilon \leq \frac{1}{3}$, under the aligned selection, majority rule (label* F *above $\frac{1}{2}$,* S *below,* I *at $\frac{1}{2}$) is anonymous, independent, valid, non-imposed and non-null.*

Proof. Anonymity, independence. The rule depends only on shares.

Validity. In $\sigma(\mu)$, p_2 beats p_1: the citizens closer to p_2 than to p_1 are those whose projection exceeds t_1, of mass $\frac{2}{3} > \frac{1}{2}$. Likewise p_2 beats p_3 with $\frac{2}{3}$. The labels of the triple contain an F and an S, since p_2 is strictly first, so they are valid (Lemma 5), whatever the label of $\{p_1, p_3\}$.

Non-nullness. $p_2 \succ p_1$ strictly.

Non-imposition. Let a, b be offered together in $\sigma(\mu)$. Let R be the reflection in the bisector of a and b, and $\mu' = R\mu$, which charges no line. I claim $\sigma(\mu') = R\sigma(\mu)$. Since $a, b \in L(\mu)$, the bisector is orthogonal to $L(\mu)$, and R maps $L(\mu)$ to itself. The mean of μ' is the image of that of μ. The covariance of μ' is $R\Sigma R^\top$, with the same principal eigenvector up to sign, or isotropic if Σ is; in the latter case $L(\mu)$ is horizontal and R is a reflection with vertical axis, which preserves horizontal lines. So $L(\mu') = L(\mu)$. On this line R reverses the orientation: $F'(t) = 1 - F(Rt)$, so $\{F' = 1/3\} = R\{F = 2/3\}$, and $t'_1 = Rt_2$, $t'_2 = Rt_1$. The formulas for p_1, p_2, p_3 are affine combinations; they give $p'_1 = Rp_3$, $p'_2 = Rp_2$ and $p'_3 = Rp_1$. Hence $a = Rb$ and $b = Ra$ are offered together in $\sigma(\mu')$, and $s_{ab}(\mu') = \mu\{y : |Ry - a| < |Ry - b|\} = \mu\{y : |y - b| < |y - a|\} = 1 - s_{ab}(\mu)$. If majority puts $a \succ b$ at μ, it puts $b \succ a$ at μ'; if it puts $a \sim b$, both weak directions occur. Each ordered pair is thus ranked weakly above at least once. □

For μ uniform on the square $[-1, 1]^2$, the covariance is isotropic, the line is horizontal, $t_1 = -\frac{1}{3}$, $t_2 = \frac{1}{3}$, and the offer is $\{-\frac{2}{3}, 0, \frac{2}{3}\}$ on the horizontal axis. The centre party beats each of the others by $\frac{2}{3}$ to $\frac{1}{3}$. For the pair $\{-\frac{2}{3}, 0\}$, the reflection with axis $x = -\frac{1}{3}$ gives μ' uniform on $[-\frac{5}{3}, \frac{1}{3}] \times [-1, 1]$, with offer $\{-\frac{4}{3}, -\frac{2}{3}, 0\}$: now $-\frac{2}{3}$ is in the centre and beats 0.

The impossibility proof does not apply because the selected offers are aligned. On three aligned points citizens produce only four orders: second choices are not free, and the centre cell always contains a Condorcet winner. The selection does not recognize obvious clusters: for three clusters at the vertices of a triangle, it picks three aligned points. Equivariance does not exclude it, since it is equivariant under the reflections used in the proof.

Continuity is not enough. Equip $\mathcal{M}_0$ with weak convergence and offers with the Hausdorff distance.

Proposition S7 (A continuous perverse selection)**.** *Let $k = 3$, $0 < \varepsilon < \frac{1}{3}$ and $\delta = \frac{1}{2}(\frac{1}{3} - \varepsilon)$. For $\mu \in \mathcal{M}_0$, let Q_x (resp. Q_y) be the quantile function of the projection of μ on the horizontal (resp. vertical) axis, and*

$$t_1 = \tfrac{1}{2\delta}\int_{1/3-\delta}^{1/3+\delta} Q_x, \qquad t_2 = \tfrac{1}{2\delta}\int_{2/3-\delta}^{2/3+\delta} Q_x, \qquad y_0 = \tfrac{1}{2\delta}\int_{1/2-\delta}^{1/2+\delta} Q_y.$$

*Let $\sigma(\mu) = \{p_1, p_2, p_3\}$, on the line $y = y_0$, with abscissae $\frac{3t_1 - t_2}{2}$, $\frac{t_1+t_2}{2}$, $\frac{3t_2-t_1}{2}$. Then σ is defined on all of $\mathcal{M}_0$, continuous, and $\sigma(\mu) \in \mathrm{Eq}(\mu)$. Under this selection, majority rule (*I *at $\frac{1}{2}$) is anonymous, independent, valid, non-imposed and non-null.*

Proof. Equilibrium. μ charges no vertical line, so the distribution function F of the abscissa is continuous and $F(Q_x(u)) = u$ for $u \in (0,1)$. Since Q_x is non-decreasing, t_1 lies between $Q_x(\frac{1}{3} - \delta)$ and $Q_x(\frac{1}{3} + \delta)$, so $F(t_1) \in [\frac{1}{3} - \delta, \frac{1}{3} + \delta]$; likewise $F(t_2) \in [\frac{2}{3} - \delta, \frac{2}{3} + \delta]$. Since $\delta < \frac{1}{6}$, $t_1 < t_2$. The cells of $\sigma(\mu)$ are the vertical strips cut at t_1 and t_2, the midpoints of p_1p_2 and p_2p_3. Their masses are at least $\frac{1}{3} - \delta$, $\frac{1}{3} - 2\delta = \varepsilon$ and $\frac{1}{3} - \delta$. No moment condition is needed: Q_x is bounded on $[\frac{1}{3} - \delta, \frac{2}{3} + \delta] \subset (0,1)$.

Continuity. If $\mu_n \to \mu$ weakly, the projections converge weakly, and $Q_x^{(n)}(u) \to Q_x(u)$ at every continuity point of Q_x, hence almost everywhere. On $[\frac{1}{3} - \delta, \frac{1}{3} + \delta]$ the functions $Q_x^{(n)}$ are uniformly bounded for n large, since $Q_x^{(n)}(\frac{1}{3} - \delta)$ and $Q_x^{(n)}(\frac{1}{3} + \delta)$ have finite lower and upper limits. By dominated convergence, $t_1(\mu_n) \to t_1(\mu)$. The same holds for t_2 and y_0.

Validity. p_2 beats p_1 with the share $1 - F(t_1) \geq \frac{2}{3} - \delta > \frac{1}{2}$, and beats p_3 with $F(t_2) > \frac{1}{2}$. It is strictly first, so the labels of the triple are valid.

Non-imposition. Let p_i, p_j be offered together, and R the reflection with vertical axis $x = c$, where c is the abscissa of their midpoint. The projection of $R\mu$ has quantile function $u \mapsto 2c - Q_x(1-u)$ almost everywhere, so $t_1(R\mu) = 2c - t_2(\mu)$ and $t_2(R\mu) = 2c - t_1(\mu)$; y_0 does not change. Hence $\sigma(R\mu) = R\sigma(\mu)$, and R exchanges p_i and p_j; for $\{p_1, p_3\}$, c is the abscissa of p_2. The share of p_i against p_j goes from s to $1 - s$. If majority puts $p_i \succ p_j$ at μ, it puts $p_j \succ p_i$ at $R\mu$; if it puts I, both weak directions occur. Non-nullness: $p_2 \succ p_1$. □

Remark S5. (1) The selection does not recognize obvious clusters: for three clusters at the vertices of a triangle, it picks three aligned points. So continuity does not imply the cluster property, and it is the latter that matters. Whether a selection can recognize clusters while being continuous is not shown here. (2) For $\varepsilon = \frac{1}{3}$ the strips must have mass exactly $\frac{1}{3}$; take $t_1 = Q_x(\frac{1}{3})$ and $t_2 = Q_x(\frac{2}{3})$. This is continuous on the subdomain of the μ whose projection has a distribution function strictly increasing on its support, which is connected; this subdomain is stable under the reflections used, and the conclusion holds on it. (3) No selection can be equivariant under *all* isometries on a domain that contains a rotation-invariant distribution, such as the uniform distribution on a disc: a finite set of $k \geq 2$ points is not invariant under all rotations about a point. Partial equivariance is the most one can ask.

Collinear equilibrium offers

In the mean-of-members model, the aligned selection is not defined everywhere.

Proposition S8 (No collinear equilibrium offer)**.** *Let μ be the uniform distribution on a triangle $V_1V_2V_3$ that is not isosceles, and $k \geq 3$. No set of k aligned points is an equilibrium offer for μ.*

Proof. Reduction. Let O be aligned on a line of direction e; x denotes the coordinate along e, y the orthogonal coordinate, and $\bar{y}$ that of the barycentre G of μ. The cells of O are strips $\{t_{i-1} < x < t_i\}$. If each platform is the mean of its cell, all cells have the same mean in y, equal

to the ordinate y_O of the line; weighting by the masses, $y_O = \bar{y}$. Let $\varphi(t) = \int_{\{x<t\}}(y-\bar{y})\,d\mu$. Then $\varphi(t_i) = 0$ at each boundary. Since the cells have positive mass, the $k-1 \geq 2$ boundaries are distinct and lie in the open interval $(x_{\min}, x_{\max})$ of the abscissae of the vertices. It suffices to show that φ has at most one zero in this interval.

The function φ. Let $f(s)$ be the length of the chord $\{x = s\}$ and $g(s)$ the ordinate of its midpoint minus $\bar{y}$. Then $\varphi' = \kappa f g$ with $\kappa > 0$, $f > 0$ on $(x_{\min}, x_{\max})$, and φ vanishes at both ends. If φ had two interior zeros, Rolle's theorem would give three distinct zeros of g in the open interval.

The zeros of g. Order the vertices by abscissa, $x_1 \leq x_2 \leq x_3$. On $[x_1, x_2]$ the chord joins the sides V_1V_2 and V_1V_3; its endpoints depend affinely on s and meet at V_1, so its midpoint describes a segment from V_1, and g is affine. The same holds on $[x_2, x_3]$, with V_3. Suppose g vanishes on the whole first piece. The line $\{y = \bar{y}\}$, which is parallel to e, then cuts every chord at its midpoint, and it passes through V_1, since $g(x_1) = 0$ gives $y(V_1) = \bar{y}$. It is therefore an axis of symmetry of the lines V_1V_2 and V_1V_3, that is, the bisector of the angle at V_1; and it passes through G. The bisector at V_1 would be the median, and the triangle would be isosceles at V_1. The same holds for the second piece and V_3. So each piece carries at most one zero, and g has at most two. If $x_1 = x_2$ or $x_2 = x_3$, there is only one piece. □

A heuristic count explains the result. An aligned configuration of k points has $k+2$ unknowns (the direction, the position of the line, and the k points on it) and must satisfy $2k$ equations (two coordinates per centroid): there are $k-2 \geq 1$ equations too many. Symmetric distributions escape, because the orthogonal conditions hold automatically on an axis. For $k = 2$, every distribution has an aligned equilibrium offer. The result uses that all cells have positive mass, and it is not uniform: if the mass of one cell may tend to 0, numerical computation finds approximate aligned equilibrium offers as good as one likes (an aligned equilibrium offer with two points, plus a point in a corner with a tiny cell). Any version of the aligned selection obtained by a limit must take this into account.

Proposition S9 (A perverse selection on the symmetric domain)**.** *Let $k \geq 3$ and let $\mathcal{M}_{\text{sym}}$ be the set of distributions of $\mathcal{M}_0$, with a finite second moment, that have at least one axis of symmetry. For $\mu \in \mathcal{M}_{\text{sym}}$, choose an axis $\ell(\mu)$ and a set of k points of $\ell(\mu)$ that minimizes the mean squared distance to the projection of μ on $\ell(\mu)$; this is $\sigma(\mu)$. Then $\sigma(\mu)$ is an equilibrium offer. Under this selection, the rule "$p \succ q$ if $s_{pq} > \frac{1}{2}$, or if $s_{pq} = \frac{1}{2}$ and $p <_{\text{lex}} q$" is anonymous, independent, valid, non-imposed and non-null.*

Proof. Equilibrium. The projection ν of μ on $\ell(\mu)$ has no atom, since μ charges no line, and has a finite second moment. Such a set of k points exists; this is classical for a distribution with a finite second moment (Graf and Luschgy, 2000, Section 4); Lemma S24 does not give existence. By the argument of Lemma S24 in dimension one, this set has k distinct points and cells of positive mass, and each point is the mean of its cell for ν. In the plane, the cells are strips orthogonal to the axis. Each strip is symmetric with respect to the axis, so its mean lies on the axis, and it is the chosen point.

Validity. The offers are aligned, and the rule is the one of Proposition S4 with the lexicographic orientation; it gives a strict total order on each offer, whatever the distribution.

Non-imposition. Let $a \neq b$ be two points, on a line ℓ_0. Take k small discs centred on ℓ_0, with pairwise distinct masses, at points $z_1, \ldots, z_k$, with $a = z_1$, $b = z_2$ and the others beyond b. The distribution is symmetric with respect to ℓ_0 and to no other line: a reflection that preserves it maps each disc to the disc of the same mass, hence fixes each disc, hence passes through all the centres, hence is ℓ_0. So $\ell(\mu) = \ell_0$. For small enough discs, the unique optimal set of points for the projection is $\{z_i\}$, by Lemma S23, whose proof holds in dimension one. The share of a against b is the mass of the disc at a; taken below $\frac{1}{2}$, it gives $b \succ a$. With a, b

last, $a = z_{k-1}$, $b = z_k$ and the others before a, the share of a is 1 minus the mass of the disc at b, above $\frac{1}{2}$: $a \succ b$. Non-nullness is immediate. □

Whether a perverse selection exists on the whole mean-of-members model is open. Such a selection would have to pick a non-aligned offer for the uniform distribution on a non-isosceles triangle (Proposition S8), and the rule would have to remain valid there while being tied by independence to the labels obtained elsewhere. One possibility is an equilibrium offer where some cell has mass above $\frac{1}{2}$: its party is then a Condorcet winner, and majority rule is valid for $k = 3$. I do not know whether every distribution has such an offer.

Minimum size under a selection that recognizes clusters

Under a selection, a new difficulty appears at the two extreme shares. A clustered distribution produces $s_{ab} = \varepsilon$ only if all the members of c prefer b to a; with clusters whose barycentre is the platform, this requires c to be closer to b than to a. Likewise $s_{ab} = 1 - \varepsilon$ requires c closer to a. A single triple therefore cannot produce crosses through both extremes. I bypass this by asking non-imposition to be visible away from the extremes. In this subsection $D_{ab} \subset [\varepsilon, 1-\varepsilon]$, since every cell has mass at least ε, and

$$J = (\varepsilon, 1 - \varepsilon).$$

Definition S3 (Interior non-imposition)**.** A rule is *non-imposed in the interior* if, for each offered pair, ℓ_{ab} is neither constantly F nor constantly S on $D_{ab} \cap J$.

Interior non-imposition implies non-imposition, and weak robust non-imposition implies it (Definition S5(i)), since an interval of positive length in $D_{ab} \subset [\varepsilon, 1-\varepsilon]$ meets J in an interval.

Lemma S12 (Straddling cluster)**.** *Let Λ be a line, t a point at distance d from Λ, and $\rho > d$. There is a probability carried by $\bar{B}(t, \rho)$, with barycentre t, a weighted mixture of two uniform distributions on small discs, that charges both open half-planes bounded by Λ. If $d > 0$, there is also one that charges only the half-plane of t.*

Proof. Take an axis orthogonal to Λ, with origin on Λ, oriented towards t (any orientation if $d = 0$). Place a disc of radius r centred at the point of abscissa $-x$ and another centred at the point of abscissa y, on the normal to Λ through t, with $0 < x < \rho - d$, $d < y < d + \rho$, and r small enough that the discs lie in $\bar{B}(t, \rho)$ and do not touch Λ. With weights $q = (y-d)/(x+y)$ and $1 - q$, both in $(0, 1)$, the barycentre has abscissa $-qx + (1-q)y = d$: it is t. If $d > 0$, a disc centred at t with radius less than d gives the second claim. □

Lemma S13 (Cross at the centre)**.** *Let $k \geq 3$, $0 < \varepsilon \leq 1/k$, and a selection that recognizes obvious clusters. For every equilateral triangle $T = \{a, b, c\}$ and every $w \in J$, the vectors $(w, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, w, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, w)$ lie in V_T. In particular $J \subset D_{xy}$ for every pair.*

Proof. Sizes m are relative to T: m_t is the mass of the citizens who rank t first in T. As in Section 4, $s_{ab} = m_a + \alpha_c$, $s_{bc} = m_b + \alpha_a$, $s_{ca} = m_c + \alpha_b$, where α_c is the mass of members of c who prefer a to b, and so on. Each cluster below lies in $\bar{B}(z, \rho)$ around its platform z, has barycentre z and mass at least ε, and the platforms are distinct. Their mutual distances are bounded below by a constant $L > 0$ that does not depend on ρ: all platforms but the central party of the last case are fixed before ρ, and that party lies within ρ of ω, hence at distance at least $|\omega a| - \rho \geq |\omega a|/2$ from a, b and c for ρ small. For $\rho < L/2$, every point of a cluster is strictly closer to its own platform than to the others, so the cells are the clusters, the offer lies in $\mathrm{Eq}(\mu)$, and for $\rho < \bar{\rho}(\varepsilon, L)$ the selection picks it.

Case $\varepsilon < 1/k$. Let $w \in (\varepsilon, \frac{1}{2}]$, $j_c = \lceil (k-3)/2 \rceil$ and $j_b = \lfloor (k-3)/2 \rfloor$. Choose m_c with

$$m_c \geq (1 + j_c)\varepsilon, \qquad m_c > \tfrac{1}{2} - \varepsilon, \qquad m_c < \tfrac{1}{2}, \qquad m_c \leq 1 - (2 + j_b)\varepsilon,$$

and let $m_a = \varepsilon$ and $m_b = 1 - \varepsilon - m_c$. Such an m_c exists. First, $(1 + j_c)\varepsilon < \frac{1}{2}$, since $1 + j_c = \lceil (k-1)/2 \rceil \le k/2$ and $\varepsilon < 1/k$. Second, $(1 + j_c)\varepsilon \le 1 - (2 + j_b)\varepsilon$ is $k\varepsilon \le 1$, since $j_b + j_c = k - 3$. Third, $\frac{1}{2} - \varepsilon < 1 - (2 + j_b)\varepsilon$ is $(1 + j_b)\varepsilon < \frac{1}{2}$, which holds since $j_b \le j_c$. Fourth, $\frac{1}{2} - \varepsilon < \frac{1}{2}$. Then $m_b, m_c \in (\frac{1}{2} - \varepsilon, \frac{1}{2})$, $m_b \ge (1 + j_b)\varepsilon$ and $m_c \ge (1 + j_c)\varepsilon$. Let

$$\alpha_a = \tfrac{1}{2} - m_b \in (0, \varepsilon), \qquad \alpha_b = \tfrac{1}{2} - m_c \in (0, m_b), \qquad \alpha_c = w - \varepsilon \in (0, m_c);$$

the second holds since $m_b + m_c = 1 - \varepsilon > \frac{1}{2}$, the third since $w \le \frac{1}{2} < \varepsilon + m_c$. The shares are then $(w, \frac{1}{2}, \frac{1}{2})$. They are produced by clusters. The cluster of $t \in T$, of mass n_t, consists of two discs balanced around t, as in Lemma S25, on both sides of the line (ωt); this is possible since $t \in r_t$ for an equilateral triangle. The $k - 3$ other parties, of mass ε each, are placed far away, j_c on r_c and j_b on r_b, each with a cluster of two balanced discs on either side of the half-line; for ρ small such a cluster lies in the cell of c (resp. b) relative to T, by the argument of Lemma S4(iii). So $n_a = \varepsilon$, $n_b = m_b - j_b\varepsilon \ge \varepsilon$ and $n_c = m_c - j_c\varepsilon \ge \varepsilon$. Each cluster of members of t puts the same fraction $\alpha_t/m_t \in (0, 1)$ of its mass on the required side; both discs then have positive mass, and the barycentre is the platform. All cluster masses are at least ε. So $(w, \frac{1}{2}, \frac{1}{2}) \in V_T$ for $w \in (\varepsilon, \frac{1}{2}]$. For $w \in [\frac{1}{2}, 1 - \varepsilon)$, exchange a and b: $m_b = \varepsilon$, $m_a, m_c \in (\frac{1}{2} - \varepsilon, \frac{1}{2})$, $\alpha_c = w - m_a$, $\alpha_a = \frac{1}{2} - \varepsilon$, $\alpha_b = \frac{1}{2} - m_c$. By rotation, the three crosses pass through h and cover J in each coordinate.

Case $\varepsilon = 1/k$, k odd. All cells have mass $1/k$. The same construction works with $j_b = j_c = (k - 3)/2$: then $m_a = 1/k$, $m_b = m_c = \frac{k-1}{2k} \in (\frac{1}{2} - \frac{1}{k}, \frac{1}{2})$, $n_b = n_c = \frac{1}{k}$, and $\alpha_a = \alpha_b = \frac{1}{2k}$ and $\alpha_c = w - \frac{1}{k}$ lie in the required intervals for $w \in (\frac{1}{k}, \frac{1}{2}]$. The other half follows by exchanging a and b, the rest by rotation.

Case $\varepsilon = 1/k$, k even. Now $m_b + m_c = 1 - \frac{1}{k}$ is an odd multiple of $\frac{1}{k}$, one of the two is at least $\frac{1}{2}$, and the construction above fails. I add a *central party*. The parties are a, b, c; $j = (k - 4)/2$ far parties on r_b and as many on r_c, with clusters of two discs on either side of the half-line, as above; and a party d near ω. Its cluster consists of small discs near ω, in chosen sectors of T, and d is their barycentre; the six sectors meet at ω, so such discs exist within ρ of ω. Since the discs are near ω, they lie in the cell of d, far from a, b, c. Each cluster has mass $1/k$; the offer is an equilibrium and the selection picks it.

Let $v = (v_1, v_2, v_3)$ be the mean share vector of the members of d. With four discs in the sectors *bac*, *cba*, *abc*, *cab*, of weights $\frac{1-v_1}{2}$, $\frac{1-v_1}{2}$, $\frac{v_1}{2}$, $\frac{v_1}{2}$, one gets $v = (v_1, \frac{1}{2}, \frac{1}{2})$ for every $v_1 \in [0, 1]$, since the share vectors of these sectors are $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$ and $(1, 0, 1)$. The sizes relative to T of the other members are $m_a = \frac{1}{k}$ and $m_b = m_c = \frac{1+j}{k} = \frac{k-2}{2k}$, and

$$s_{ab} = \tfrac{1}{k} + \alpha_c + \tfrac{v_1}{k}, \qquad s_{bc} = m_b + \alpha_a + \tfrac{1}{2k}, \qquad s_{ca} = m_c + \alpha_b + \tfrac{1}{2k},$$

with $\alpha_a \in (0, \frac{1}{k})$, $\alpha_b \in (0, m_b)$, $\alpha_c \in (0, m_c)$ free. Take $\alpha_a = \alpha_b = \frac{1}{2k}$: then $s_{bc} = s_{ca} = \frac{1}{2}$. For $w \in (\frac{1}{k}, \frac{1}{2})$, take $v_1 = 0$ and $\alpha_c = w - \frac{1}{k} \in (0, \frac{k-2}{2k})$; for $w = \frac{1}{2}$, take $v_1 > 0$ small and $\alpha_c = \frac{1}{2} - \frac{1+v_1}{k}$. This gives $(w, \frac{1}{2}, \frac{1}{2})$ for $w \in (\frac{1}{k}, \frac{1}{2}]$. The rest follows by exchanging a and b and by rotation. □

Lemma S14 (Extremes)**.** *Under the same hypotheses, for every pair $\{a, b\}$ there are a point c and a selected distribution, whose offer contains $T = \{a, b, c\}$, with shares $s_{ab} = \varepsilon$, $s_{bc} \in J$ and $s_{ca} \in J$. The same holds with $s_{ab} = 1 - \varepsilon$.*

Proof. Start from the equilateral triangle on ab and move its apex by a small vector towards b; this gives c, strictly closer to b than to a. Let d_a be the distance from a to the bisector of $\{b, c\}$, and d_b the distance from b to the bisector of $\{c, a\}$. Both vanish for the equilateral triangle, so they are as small as one likes for a small move. For a small move, the platforms, including the far parties below, stay at least $L = |ab|/2$ apart. Fix $\rho < \bar{\rho}(\varepsilon, L)$, then a move with $d_a, d_b < \rho$.

Clusters: at a, of mass ε, straddling the bisector of $\{b,c\}$ (Lemma S12); at b, of mass $n_b \geq \varepsilon$, straddling the bisector of $\{c,a\}$; at c, of mass $n_c \geq \varepsilon$, a small disc on the side of b; the $k-3$ other parties, of mass ε, as small far discs in the cell of c relative to T, on the side of b of the bisector of $\{a,b\}$, so that their order on T is $c\,b\,a$. Only the members of a prefer a to b: $s_{ab} = \varepsilon$. Next $s_{bc} = n_b + \alpha_a$ with $\alpha_a \in (0,\varepsilon)$, so $s_{bc} \in (\varepsilon, 1-\varepsilon)$, since $n_b + \varepsilon \leq 1-(k-2)\varepsilon \leq 1-\varepsilon$. Finally $s_{ca} = m_c + \alpha_b$, where $m_c = 1-\varepsilon-n_b \geq \varepsilon$ and $\alpha_b \in (0, n_b)$, so $s_{ca} \in (m_c, 1-\varepsilon) \subset J$. The offer is a clustered equilibrium, and the selection picks it. For $1-\varepsilon$, move the apex towards a and exchange the roles. □

Theorem S7 (Minimum size under selection, $0 < \varepsilon \leq 1/k$)**.** *In the minimum-size model with $k \geq 3$ parties, $0 < \varepsilon \leq 1/k$, and a selection that recognizes obvious clusters, no rule satisfies anonymity, independence, interior non-imposition and non-nullness.*

Proof. Let $\{a,b\}$ be offered and T the equilateral triangle on ab. By Lemma S13, V_T contains the three crosses through h with arms J. Interior non-imposition says that each label is neither constantly F nor constantly S on J. Lemma S1, with $J_i = J$, gives $\ell = \mathsf{I}$ on J for the three pairs of T, hence for every pair. By Lemma S14, some produced point is (ε, y, z) with $y, z \in J$; its labels are $(\ell_{ab}(\varepsilon), \mathsf{I}, \mathsf{I})$, and validity forces $\ell_{ab}(\varepsilon) = \mathsf{I}$. The same holds at $1-\varepsilon$. All labels are I. □

Lemma S14 uses *nearly* isosceles clusters, which Definition S4 allows through Lemma S12. For $k = 4$ and $\varepsilon = \frac{1}{4}$, the central party is needed. With the far parties placed on r_b or r_c as in the case $\varepsilon < 1/k$, the only possible sizes relative to T are $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and its permutations, and the open boxes $(\frac{1}{4}, \frac{3}{4}) \times (\frac{1}{4}, \frac{1}{2}) \times (\frac{1}{2}, \frac{3}{4})$, and so on, are disjoint and miss h. With d near the centre and $v = (0, \frac{1}{2}, \frac{1}{2})$, half of its members in bac and half in cba, $\alpha_a = \alpha_b = \frac{1}{8}$ and $\alpha_c = w - \frac{1}{4}$ give $(w, \frac{1}{2}, \frac{1}{2})$ for every $w \in (\frac{1}{4}, \frac{1}{2})$.

Theorem S8 (A tie at one half suffices)**.** *Under the hypotheses of Theorem S7, if $\ell_{xy}(\frac{1}{2}) = \mathsf{I}$ for every offered pair, every rule that satisfies anonymity, independence and validity is null. No non-imposition is used.*

Proof. Let T be equilateral. At h the labels are $(\mathsf{I}, \mathsf{I}, \mathsf{I})$. For $w \in J$, the product $\{\frac{1}{2}, w\} \times \{\frac{1}{2}\} \times \{\frac{1}{2}\}$ lies in V_T (Lemma S13). By Lemma 2, a rule valid on this product is either null or constant and strict on two factors; the two factors reduced to $\{\frac{1}{2}\}$ carry I, so it is null: $\ell_{ab}(w) = \mathsf{I}$. Hence $\ell = \mathsf{I}$ on J for every pair. Lemma S14 concludes as before. □

A rule that is neutral between the two parties of a pair puts I at the share $\frac{1}{2}$. Under a selection that recognizes clusters, such a rule is null even without non-imposition. A rule that escapes must take sides at an exact tie.

Wilson's non-imposition alone. The pair $\{a,b\}$ is *rigid*, oriented $a \to b$, if $\ell_{ab} = \mathsf{F}$ on $D_{ab} \cap J$: a beats b at every interior share. Interior non-imposition says exactly that no pair is rigid in either direction. If $a \to b$ is rigid, Wilson's non-imposition asks for an extreme share at which $b \succeq a$:

(W1) $\ell_{ab}(\varepsilon) \in \{\mathsf{S}, \mathsf{I}\}$: when a has only the minimal share, b does at least as well;

(W2) $\ell_{ab}(1-\varepsilon) \in \{\mathsf{S}, \mathsf{I}\}$: when b has only the minimal share, b does at least as well.

Proposition S10 (Reduction to rigid pairs)**.** *Under the hypotheses of Theorem S7, a rule that satisfies anonymity, independence, validity and non-nullness has at least one rigid pair. More precisely, for every equilateral triangle T, either the three pairs of T are I on J, or one of them is rigid.*

Proof. The second sentence is Lemma S1 with $J_i = J$: its hypothesis is exactly that none of the three pairs is rigid. If no pair were rigid, every pair would be I on J, and Lemma S14 would make the rule null. Non-imposition is not used. □

In the rest of this subsection $k = 3$ and $0 < \varepsilon < \frac{1}{3}$. I use exact clusters: tight clusters whose centre lies on a bisector straddle it with two balanced discs, and clusters whose centre lies off all bisectors are single small discs.

Lemma S15 (Equilateral triangle)**.** *Let $T = \{a, b, c\}$ be equilateral. If $a \to b$ is rigid, then $c \to b$ is rigid ($\ell_{bc} = \mathsf{S}$ on J) or $a \to c$ is rigid ($\ell_{ca} = \mathsf{S}$ on J).*

Proof. Every pair $(y, z) \in J^2$ is produced in coordinates 2 and 3, with a first coordinate in J. The shares are $(n_a+\alpha_c,\ n_b+\alpha_a,\ n_c+\alpha_b)$ with $n_t \geq \varepsilon$, $\sum n_t = 1$ and $\alpha_t \in (0, n_t)$. Let $U_a = 1-z$, $U_b = y$ and $U_c = \min(z, 1-y)$. Each exceeds ε, and $U_a+U_b+U_c = \min(1+y, 2-z) > 1 \geq 3\varepsilon$. So there are $n_t \in [\varepsilon, U_t)$ of sum 1. Then $\alpha_a = y - n_b \in (0, n_a)$, since $y < 1 - n_c$, and $\alpha_b = z - n_c \in (0, n_b)$, since $z < 1 - n_a$. The first coordinate lies in $(n_a, 1 - n_b) \subset J$.

Conclusion. At each of these points the label of ab is F, and validity requires an S among $\ell_{bc}(y)$ and $\ell_{ca}(z)$. If $\ell_{bc}(y_0) \neq \mathsf{S}$ and $\ell_{ca}(z_0) \neq \mathsf{S}$, the point with coordinates (y_0, z_0) would be invalid. So $\ell_{bc} = \mathsf{S}$ on J, or $\ell_{ca} = \mathsf{S}$ on J. □

Lemma S16 (Propagation, type (W1))**.** *Let $a \to b$ be rigid, with (W1). For every c with $|bc| = |ab| < |ac|$, so that b sees a and c under an angle in $(60°, 180°]$:*

- *either $c \to a$ is rigid and $\ell_{bc}(w) = \mathsf{S}$ ($c \succ b$) for every $w \in (2\varepsilon, 1 - \varepsilon]$;*
- *or $a \to c$ is rigid and $\ell_{bc}(w) = \mathsf{F}$ ($b \succ c$) for every $w \in (2\varepsilon, 1 - \varepsilon]$.*

Proof. Produced shares. The triangle is isosceles at b. The members of a prefer b to c, since $|ab| < |ac|$; those of c prefer b to a; those of b, whose cluster straddles the bisector of $\{a, c\}$, split: a mass $\alpha \in (0, n_b)$ prefers c to a. The shares are $(n_a,\ 1 - n_c,\ n_c + \alpha)$.

Interior shares of a. Fix $u = n_c \in [\varepsilon, 1 - 2\varepsilon)$ and $z \in (u, 1 - \varepsilon)$. Choose $n_a \in (\varepsilon, \min(1 - z, 1 - u - \varepsilon))$ and $\alpha = z - u$; this produces $(n_a, 1 - u, z)$ with $n_a \in J$, with labels $(\mathsf{F}, \ell_{bc}(1 - u), \ell_{ca}(z))$. An S is needed: $\ell_{bc}(1 - u) = \mathsf{S}$ or $\ell_{ca}(z) = \mathsf{S}$.

Minimal share of a. With $n_a = \varepsilon$, this produces $(\varepsilon, 1 - u, z)$, with labels $(E, \ell_{bc}(1 - u), \ell_{ca}(z))$, where $E = \ell_{ab}(\varepsilon) \in \{\mathsf{S}, \mathsf{I}\}$. If $E = \mathsf{S}$, an F is needed among the two others. If $E = \mathsf{I}$, the two others are (I, I) or strict and opposite, and (I, I) is excluded by the previous paragraph.

Combination. In both cases $(\ell_{bc}(1 - u), \ell_{ca}(z)) \in \{(\mathsf{S}, \mathsf{F}), (\mathsf{F}, \mathsf{S})\}$, for every $u \in [\varepsilon, 1 - 2\varepsilon)$ and every $z \in (u, 1 - \varepsilon)$. With $u = \varepsilon$, ℓ_{ca} is constant and strict on J, and the pair $\{c, a\}$ is rigid. If $\ell_{ca} = \mathsf{F}$ on J ($c \to a$), then $\ell_{bc}(1 - u) = \mathsf{S}$ for every u; if $\ell_{ca} = \mathsf{S}$ ($a \to c$), then $\ell_{bc}(1 - u) = \mathsf{F}$. Finally, $1 - u$ ranges over $(2\varepsilon, 1 - \varepsilon]$. □

Lemma S17 (Propagation, type (W2))**.** *Let $a \to b$ be rigid, with (W2). For every c with $|ac| = |ab| < |bc|$:*

- *either $b \to c$ is rigid and $\ell_{ca}(v) = \mathsf{S}$ ($a \succ c$) for every $v \in [\varepsilon, 1 - 2\varepsilon)$;*
- *or $c \to b$ is rigid and $\ell_{ca}(v) = \mathsf{F}$ ($c \succ a$) for every $v \in [\varepsilon, 1 - 2\varepsilon)$.*

Proof. The triangle is isosceles at a. The members of b prefer a to c, those of c prefer a to b, and those of a split: a mass $\alpha \in (0, n_a)$ prefers b to c. The shares are $(1 - n_b,\ n_b + \alpha,\ n_c)$. Fix $v = n_c \in [\varepsilon, 1 - 2\varepsilon)$ and $y \in (\varepsilon, 1 - v)$. With $n_b \in (\varepsilon, \min(y, 1 - v - \varepsilon))$, this produces $(1 - n_b, y, v)$, whose first label is F: so $\ell_{bc}(y) = \mathsf{S}$ or $\ell_{ca}(v) = \mathsf{S}$. With $n_b = \varepsilon$, it produces $(1 - \varepsilon, y, v)$, whose first label lies in $\{\mathsf{S}, \mathsf{I}\}$ by (W2). As in Lemma S16, $(\ell_{bc}(y), \ell_{ca}(v)) \in \{(\mathsf{S}, \mathsf{F}), (\mathsf{F}, \mathsf{S})\}$. With $v = \varepsilon$, y ranges over J: $\{b, c\}$ is rigid, and the sign of $\ell_{ca}(v)$ follows. □

The mechanism, step by step. (1) A rigid pair $a \to b$ must yield, for Wilson, at an extreme share. (2) In a triangle isosceles at b, the share s_{ab} moves from the interior to the value ε while the two other shares stay put. (3) The labels of the two other pairs must therefore make both $(\mathsf{F}, \cdot, \cdot)$ and $(\mathsf{S} \text{ or } \mathsf{I}, \cdot, \cdot)$ valid: they are strict and opposite. (4) Since the share of c against a ranges over all of J, the pair $\{a, c\}$ becomes rigid. Rigid pairs propagate along an arc of 240° of the circle centred at b.

Proposition S11 (Fully rigid rules)**.** *Let $k = 3$, $0 < \varepsilon < \frac{1}{3}$, and a selection that recognizes obvious clusters. No rule that satisfies anonymity, independence and validity satisfies both:*

(A) every pair is rigid;

(B) for every pair, rigid $x \to y$, (W1) holds: $\ell_{xy}(\varepsilon) \in \{\mathsf{S}, \mathsf{I}\}$.

The same holds with (W2) in place of (W1) in (B).

Proof. A total order. By (A), write $x \succ^* y$ if $x \to y$. For three distinct points, tight clusters of masses above ε give interior shares: the share of x against y lies between $n_x > \varepsilon$ and $n_x + n_z = 1 - n_y < 1 - \varepsilon$. The labels there are those of $\succ^*$, and validity forbids a cycle. So $\succ^*$ is a tournament on the plane without 3-cycles, that is, a strict total order.

A cycle. Let x, y, z satisfy $|zx| < |yz| < |xy|$ and $x \succ^* y \succ^* z$. Take tight clusters, none centred on a bisector, with masses $n_x = \varepsilon$ and $n_y, n_z > \varepsilon$. The orders are $x\,z\,y$ for the members of x, since $|xz| < |xy|$; $y\,z\,x$ for those of y; and $z\,x\,y$ for those of z. The shares are $s_{xz} = \varepsilon$, $s_{xy} = \varepsilon + n_z \in J$ and $s_{yz} = n_y \in J$. By (B) applied to $x \to z$, $z \succeq x$. By (A), $x \succ y$ and $y \succ z$. This is invalid. It therefore suffices to find such a triangle for every total order.

Every total order has one. Suppose not. For $u \succ^* v$, let $R(u, v) = \{w : |uw| < |vw| < |uv|\}$. If $w \in R(u, v)$, the triangle (u, v, w) satisfies $|wu| < |vw| < |uv|$; since there is no triangle of the kind sought, $w \succ^* v$. So $R(u, v) \subset U(v) = \{w : w \succ^* v\}$ whenever $u \in U(v)$.

(i) *The geometry of R.* If $r = |uv|$, $R(u, v)$ contains every w with $|vw| = s \in (r/\sqrt{2}, r)$ and an angle $\theta \leq \pi/4$ between vu and vw. Indeed $|uw|^2 = r^2 + s^2 - 2rs\cos\theta \leq r^2 + s^2 - \sqrt{2}\,rs < s^2$, since $r < \sqrt{2}s$.

(ii) *A neighbourhood.* Let $u \in U(v)$, $r = |uv|$ and $\gamma = 0.99$. Let u_j be the point at distance $\gamma^j r$ from v, in the direction of vu rotated by $j\pi/8$, for $j = 0, \ldots, 16$. By (i), $u_{j+1} \in R(u_j, v)$, so by induction $u_j \in U(v)$. By (i) again, $U(v)$ contains the sectors of half-angle $\pi/4$ on either side of vu_j, at distances in $(\gamma^j r/\sqrt{2}, \gamma^j r)$. These sectors cover all directions, and their intervals of distances all contain $(r/\sqrt{2}, \gamma^{16} r)$, which is non-empty since $\gamma^{16} > 0.85 > 1/\sqrt{2}$. So $U(v)$ contains every circle centred at v with radius $s \in (r/\sqrt{2}, \gamma^{16} r)$. Starting again from a point of such a circle, it contains the circles with radius in $(s/\sqrt{2}, \gamma^{16} s)$. Since $\gamma^{32} > 0.72 > 1/\sqrt{2}$, these intervals overlap, and by induction $U(v) \supset B(v, \gamma^{16} r) \setminus \{v\}$.

(iii) *Baire.* For $n \geq 1$, let $A_n = \{v : B(v, 1/n) \setminus \{v\} \subset U(v)\}$. By (ii), every point that is not the largest for $\succ^*$ lies in some A_n. The plane is therefore the union of the A_n and of at most one point. By Baire's theorem, in the form "a complete metric space is not a countable union of nowhere dense sets", some A_n is dense in an open ball Ω. Let $v \in A_n \cap \Omega$; the open set $\Omega \cap B(v, 1/n) \setminus \{v\}$ contains some $v' \in A_n$. Then $v' \succ^* v$, since $v \in A_n$, and $v \succ^* v'$, since $v' \in A_n$: a contradiction.

The case (W2). The reversed rule, in which every weak order is replaced by its inverse, satisfies anonymity, independence and validity, and has the same rigid pairs, with opposite directions. If $x \to y$ satisfies (W2) for the rule, then $y \to x$ satisfies (W1) for the reversed rule. The argument above applies. □

Corollary S4 (Monotone rules)**.** *Call a rule* monotone *if $\ell_{xy}(w)$ is non-decreasing in w for the order $\mathsf{S} < \mathsf{I} < \mathsf{F}$. Under the hypotheses of Proposition S11, no monotone, valid and non-imposed rule has all its pairs rigid.*

Proof. For a rigid pair $x \to y$, (W2) would require $\ell_{xy}(1-\varepsilon) < \mathsf{F} = \ell_{xy}(w)$ for $w < 1-\varepsilon$, which monotonicity excludes. Non-imposition therefore gives (W1) for every pair, and Proposition S11 applies. □

Let $\succ^*$ be the lexicographic order, the smaller point being preferred, and take the rule "the party that has only the share ε against the other loses; otherwise $\succ^*$ decides". It satisfies (A) and (B). With $x = (0,0)$, $y = (0.3, 5)$, $z = (0.6, 1.2)$: $|zx| \approx 1.34 < |yz| \approx 3.81 < |xy| \approx 5.01$, and $x \succ^* y \succ^* z$. With $n_x = 0.1$, $n_y = n_z = 0.45$ and $\varepsilon = 0.1$, z beats x (share 0.1), x beats y (share 0.55), and y beats z (share 0.45, lexicographic order): a cycle.

Remark S6 (What is open). (1) *Wilson's non-imposition alone, in general.* A rule that escapes has rigid pairs (Proposition S10); they propagate (Lemmas S15–S17); and they cannot all be rigid with witnesses of the same type (Proposition S11). The rules in which some pairs are rigid and others are not, or whose Wilson witnesses mix (W1) and (W2), are not covered. When all pairs are rigid, the proof of Proposition S11 gives a necessary condition on $\succ^*$ alone: for every pair $u \succ^* v$, the interval (v, u) of the order avoids one of the two open regions $\{w : |uv| < |vw| < |uw|\}$ and $\{w : |uv| < |uw| < |vw|\}$, according to the type of the witness. I do not know whether a total order of the plane can satisfy this; the Baire argument does not apply as such, because the type may change from one pair to another. No counterexample is known either. (2) $k \geq 4$ *under Wilson's non-imposition alone.* Lemmas S15–S17 are written for $k = 3$; Proposition S10 and Theorems S7 and S8 hold for every k. (3) A perverse selection defined on the whole mean-of-members model, discussed after Proposition S9.

S1.4 Crowded offers

In this subsection there is no selection. The domain consists of the pairs (x, O) with $\mu_x \in \mathcal{M}_0$, $O \in \mathcal{O}$ and every cell of mass at least ε, where $\mathcal{O}$ is a given set of offers of k distinct points and $0 < \varepsilon \leq 1/k$. Proposition 7 takes for $\mathcal{O}$ all configurations, and Remark S10 weakens this to conditions (L1)–(L3). For a non-aligned triple $T \subset O$ and $p \in O$, let Π_p be the set of strict orders π on T whose open sector meets the open cell V_p, and $e_\pi \in \{0,1\}^3$ the share vector of a citizen of order π.

Lemma S18 (The polytope of shares)**.** *The set of share vectors on T produced by the offer O is the polytope*

$$P_T(O) = \Big\{ \sum_{p \in O} \sum_{\pi \in \Pi_p} \lambda_{p\pi} e_\pi : \ \lambda \geq 0, \ \textstyle\sum_\pi \lambda_{p\pi} \geq \varepsilon \text{ for all } p, \ \sum_{p,\pi} \lambda_{p\pi} = 1 \Big\}.$$

It is convex. For a fixed offer, D_{xy} is the interval $[\varepsilon A_{xy}, 1 - \varepsilon A_{yx}]$, where A_{xy} is the number of cells contained in the closed half-plane of the points at least as close to x as to y.

Proof. A distribution that charges no line puts its mass in the open intersections $V_p \cap S_\pi$, the boundaries being null; the masses $\lambda_{p\pi}$ satisfy the constraints, and the shares are $\sum \lambda_{p\pi} e_\pi$. Conversely, discs inside these intersections produce every λ. The set is the linear image of a polyhedron. For D_{xy}: each cell contained in the closed half-plane of x gives at least ε to x; the share of x is minimal when these cells have mass ε and every other cell, which meets the open half-plane of y, puts all its mass there; likewise for the maximum. The image is an interval by convexity. □

For a triple T let $h = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $\delta^-_{xy} = \inf D_{xy}$ and $\delta^+_{xy} = \sup D_{xy}$, computed over all of $\mathcal{O}$.

Theorem S9 (The cross criterion)**.** *Suppose every offered pair belongs to a non-aligned triple $T = \{a, b, c\}$ such that, for each pair $\{x, y\}$ of T and each $\delta \in \{\delta^-_{xy}, \delta^+_{xy}\}$, some $O \in \mathcal{O}$ with $O \supset T$ has a polytope $P_T(O)$ that contains h and the point obtained by replacing the coordinate xy of h by δ. Then no rule satisfies anonymity, independence, non-imposition and non-nullness.*

Proof. By convexity (Lemma S18), the segment from h to that point lies in $P_T(O) \subset V_T$. The six segments form three crosses through h, whose arms cover $[\delta^-_{xy}, \delta^+_{xy}] \supseteq D_{xy}$. The triple is a star triple (Remark S2), and Theorem 1 concludes. □

The criterion speaks only of the possible offers: for each offer, $P_T(O)$ is computed from the incidences between cells and sectors, and, for a fixed offer O, each condition is a linear program; the quantifier "some $O \in \mathcal{O}$" ranges in general over an infinite set. The criterion follows from (L1)–(L3): the box $B(m)$ of Remark S10 contains h and the required points, with $\delta^- = \varepsilon$ and $\delta^+ = 1 - \varepsilon$. It asks neither that the additional parties arbitrate freely nor that the extreme shares be ε and $1 - \varepsilon$: if crowding prevents a pair from reaching ε in every offer, D_{xy} shrinks accordingly, and it suffices to cover what is left. Among proofs that use crosses through h in which each arm comes from a single offer, it cannot be improved. It is not the exact condition for a proof by crosses through h: $V_T = \bigcup_O P_T(O)$ is not convex, and an arm may be covered by several polytopes without lying in any one of them; the exact condition is that each arm, restricted to D_{xy}, lies in $\bigcup_O P_T(O)$. If δ^-_{xy} is not attained, it suffices that the arm covers $D_{xy} \cap [\delta^-_{xy}, \frac{1}{2}]$. The star lemma also allows a centre other than h, at the cost of one more linear program.

At the other extreme of crowding lie aligned offers.

Proposition S12 (Aligned offers)**.** *Let $k \geq 3$, $0 < \varepsilon \leq 1/k$, and let $\mathcal{O}$ be the set of all configurations of k* aligned *points. The rule "$p \succ q$ if $s_{pq} > \frac{1}{2}$, or if $s_{pq} = \frac{1}{2}$ and $p <_{\text{lex}} q$" is anonymous, independent, valid, non-imposed and non-null.*

Proof. Validity. Let $O \in \mathcal{O}$ lie on a line ℓ, oriented by $<_{\text{lex}}$; on a line, the lexicographic order is the order along the line in one direction. Let F be the distribution function of the projection of μ on ℓ; it is continuous. Let $m^- = \min\{t : F(t) \geq \frac{1}{2}\}$. For $p < q$ in O, $s_{pq} = F(\frac{p+q}{2})$, so $p \succ q$ if and only if $F(\frac{p+q}{2}) \geq \frac{1}{2}$, that is, $\frac{p+q}{2} \geq m^-$, that is, $|p - m^-| \leq |q - m^-|$; this is checked by distinguishing $p \geq m^-$, $q \leq m^-$ and $p < m^- < q$. The social relation on O ranks by increasing distance to m^-, ties going to the left. It is a strict total order.

Non-imposition. Let $p <_{\text{lex}} q$ be offered together. The offer made of p, q and $k - 2$ points beyond q on the same line, with mass ε in the cell of p and the rest spread with at least ε per cell, gives $s_{pq} = \varepsilon < \frac{1}{2}$, hence $q \succ p$. The symmetric offer, with $k - 2$ points before p and mass ε in the cell of q, gives $s_{pq} = 1 - \varepsilon > \frac{1}{2}$, hence $p \succ q$. Non-nullness is immediate. □

On a line, preferences are single-peaked and the median decides. Intermediate crowding, where for instance $k - 3$ parties gather around a and their cells stay inside one sector of each triple, can make (L2) fail: some members then have an imposed order. The example below shows that impossibility may survive.

Proposition S13 (The rhombus)**.** *Let $a = (0, 0)$, $b = (1, 0)$, $c = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $d = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, the reflection of b in (ac); let $O = \{a, b, c, d\}$, $T = \{a, b, c\}$ and $\varepsilon = \frac{1}{4}$. Then (L1) holds, (L2) fails for every choice of the small party, and $P_T(O)$ is the box $[\frac{1}{2}, \frac{3}{4}] \times [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{4}, \frac{3}{4}]$. For the fixed offer $\mathcal{O} = \{O\}$, the triple T is a star triple. So are the triples $\{a, c, d\}$, $\{a, b, d\}$ and $\{c, b, d\}$. Each of the six pairs of O therefore lies in a star triple, and no rule satisfies anonymity, independence, non-imposition and non-nullness on the fixed offer $\{O\}$.*

Proof. The cell of d. The bisector of $\{b, d\}$ is the line (ac), and that of $\{c, d\}$ is the axis $x_1 = 0$. A point x of V_d therefore satisfies $x_1 < 0$, hence $|x - a| < |x - b|$. It is also closer to d than to a, which reads $\sqrt{3}\, x_2 - x_1 > 1$; hence $x_1 < \sqrt{3} x_2$, that is, $|x - c| < |x - b|$. So b is last in the whole cell of d. Since d lies on the bisector of $\{a, c\}$, the line (bd), V_d meets both its sides: $\Pi_d = \{acb, cab\}$. For $t \in T$, $t \in V_t$ and $t \in r_t$, since the triangle is equilateral: (L1) holds, and $\Pi_a = \{abc, acb\}$, $\Pi_b = \{bac, bca\}$, $\Pi_c = \{cab, cba\}$.

(L2) fails. V_d meets only the sectors acb and cab, in which b is strictly last. On r_a, r_b and r_c, b is tied with another platform or first, so V_d meets none of the three half-lines.

The box. All cells have mass $\frac{1}{4}$, since there are four cells of mass at least $\frac{1}{4}$. Let $\alpha, \beta, \gamma, \kappa \in [0, \frac{1}{4}]$ be the masses of a in abc, of b in bca, of c in cab, and of d in cab. The members of d all prefer a to b and c to b. So $s_{ab} = \frac{1}{4} + \gamma + \frac{1}{4}$, $s_{bc} = \alpha + \frac{1}{4}$ and $s_{ca} = \frac{1}{4} + \beta + \kappa$. The four parameters are independent, which gives the box. It is a product that contains all of V_T, and D_{ab}, D_{bc}, D_{ca} are its factors, since the offer is fixed: the star lemma applies with this single product.

The triple $\{a, c, d\}$. The reflection in (ac) exchanges b and d and fixes a and c. It maps O to O and $\{a, b, c\}$ to $\{a, c, d\}$, which is therefore a star triple.

The triple $T' = \{a, b, d\}$. The shares are (s_{ab}, s_{bd}, s_{da}). The point c is the circumcentre of $\{a, b, d\}$, since $|ca| = |cb| = |cd| = 1$. It lies on the three bisectors of T', where the six sectors meet; since $c \in V_c$, which is open, V_c meets the six sectors, and Π_c contains the six strict orders. Next, a lies on the bisector of $\{b, d\}$, since $|ab| = |ad| = 1$, so $\Pi_a = \{abd, adb\}$. A point of V_b is closer to b than to c, that is, $\sqrt{3}\, x_2 < x_1$; so $\sqrt{3}\, x_2 - x_1 < 1$, which says that it is closer to a than to d: $\Pi_b = \{bad\}$. By the reflection in (ac), $\Pi_d = \{dab\}$. The share vectors are $e_{abd} = (1, 1, 0)$, $e_{adb} = (1, 0, 0)$, $e_{bad} = (0, 1, 0)$ and $e_{dab} = (1, 0, 1)$, and the convex hull of the six orders is $K = \{v \in [0, 1]^3 : \ 1 \leq v_1 + v_2 + v_3 \leq 2\}$ (Lemma 6(c)). All cells have mass $\frac{1}{4}$. Hence

$$P_{T'}(O) = \{(\tfrac{1}{2}, \tfrac{1}{4} + \alpha, \tfrac{1}{4}) + \tfrac{1}{4} v : \ \alpha \in [0, \tfrac{1}{4}], \ v \in K\},$$

and, the offer being fixed, $D_{ab} = [\frac{1}{2}, \frac{3}{4}]$, $D_{bd} = [\frac{1}{4}, \frac{3}{4}]$ and $D_{da} = [\frac{1}{4}, \frac{1}{2}]$; these are the projections of $P_{T'}(O)$, and Lemma S18 gives D_{ab} again, with $A_{ab} = 2$ and $A_{ba} = 1$. The point $h = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the three crosses through h lie in $P_{T'}(O)$:

- $(w, \frac{1}{2}, \frac{1}{2})$, $w \in [\frac{1}{2}, \frac{3}{4}]$: $v = (4w - 2, 0, 1)$ and $\alpha = \frac{1}{4}$;
- $(\frac{1}{2}, w, \frac{1}{2})$, $w \in [\frac{1}{4}, \frac{3}{4}]$: $v = (0, v_2, 1)$, with $\alpha \in [0, \frac{1}{4}]$ and $v_2 \in [0, 1]$ such that $\alpha + \frac{v_2}{4} = w - \frac{1}{4}$;
- $(\frac{1}{2}, \frac{1}{2}, w)$, $w \in [\frac{1}{4}, \frac{1}{2}]$: $v = (0, 1, 4w - 1)$ and $\alpha = 0$.

In each case $v \in K$. The arms cover D_{ab}, D_{bd} and D_{da}, so T' is a star triple (Remark S2). The reflection in (bd), the bisector of $\{a, c\}$, exchanges a and c and fixes b and d; it maps O to O and T' to $\{c, b, d\}$, which is therefore a star triple too. The triples $\{a, b, c\}$, $\{a, c, d\}$ and $\{a, b, d\}$ cover the six pairs, and Theorem 1 concludes. □

The fourth party is stuck: its members all put b last, and it cannot arbitrate. But it changes the first place between a and c, and that is enough. What matters is not that every party arbitrates, but that the produced shares form crosses. For the pair $\{b, d\}$, the key point is that c is the circumcentre of $\{a, b, d\}$: the cell of c meets the six sectors of that triple, and it alone suffices to build the crosses through h.

Remark S7 (Crowding and the existence of a rule: open). A rule exists when all offers are aligned (Proposition S12), which is extreme crowding. I do not know a non-aligned crowded offer that admits a rule. Numerical checks, which are not proofs, find none: for fixed offers of four points (forty random shapes and eight crowded ones, among them the rhombus, a fourth point at the centre, near a vertex, or cloning a vertex, three aligned points plus one, a flat kite, a nearly aligned offer) and $\varepsilon \in \{0.05, 0.1, 0.2, 0.25\}$, the criterion of Theorem S9 holds for every pair except for the aligned offer; a direct search on discretized shares finds no rule for these non-aligned offers; and for five points, a, b, c and two satellites of a inside the triangle with $\varepsilon = \frac{1}{5}$, the criterion fails for some pairs but the direct search finds no rule. This suggests the conjecture that no rule exists on a fixed non-aligned offer when every cell must have mass at least ε (without this constraint, Theorem S1 proves it).

S2 Formation by instrumental support

This appendix proves the results of Section 7 of the paper. Throughout, $T = \{a, b, c\}$ is a non-aligned triple of births with circumcentre ω, shares on T are in cyclic order, and h is the centre.

Notation. The three bisectors of T cut the plane into six open *sectors* S_π, one per strict ranking $\pi = (\pi_1\pi_2\pi_3)$ of T. A society μ has sector masses $\lambda_\pi = \mu(S_\pi)$, and $e_\pi \in \{0,1\}^3$ is the share vector of a citizen of ranking π, so that $v(\mu) = \sum_\pi \lambda_\pi e_\pi$. The set of all such sums, λ in the simplex, is $Q = \{v \in [0,1]^3 : 1 \le v_1 + v_2 + v_3 \le 2\}$. For $A \subseteq T$, $m_j(A)$ is the mass of the citizens whose favourite party in A is j, $\bar{Y}(A) = \mathbb{E}Y_A$, and $n_i(A) = \bar{Y}(A) - \bar{Y}(A \setminus i)$. For a cost $\kappa \ge 0$, $s^\kappa_{xy} = \mu\{x' : |x' - y|^2 - |x' - x|^2 > \kappa\}$ is the mass of citizens who prefer x to y by more than κ; $s^0_{xy} = s_{xy}$.

The one tool of the constructions is a citizen placed far from the parties. She ranks them by projection on her direction, and her stake in any shift of the expected outcome grows with her distance.

Lemma S19 (Far-away citizens)**.** *(i) For a unit vector d and $t > 0$, $\omega + td \in S_\pi$ if and only if $d \cdot \pi_1 > d \cdot \pi_2 > d \cdot \pi_3$. The set I_π of such directions is a non-empty open arc shorter than a half-circle, and the six arcs with the six directions that separate them make up the circle.*

(ii) For every A and $i \in A$, $G_i(x; A) = 2\,x \cdot n_i(A) + c_i(A)$ for a constant $c_i(A)$.

(iii) If $d \in I_\pi$ and $d \cdot n_i(A) > 0$, then for every $r > 0$ the disc of radius r around $\omega + td$ lies in $S_\pi \cap \{G_i(\cdot; A) > \kappa\}$ for all t large enough.

(iv) Given sector masses λ and a finite list of pieces $(\pi, mass, d)$ with $d \in I_\pi$ and masses summing to λ_π in each sector, the society uniform on small discs around the points $\omega + td$, t large, charges no line and has sector masses λ.

Proof. (i) $|\omega + td - z|^2 = R_T^2 + t^2 - 2t\,d \cdot (z - \omega)$ with R_T the circumradius, so the ranking of $\omega + td$ is the order of $d \cdot z$ for every $t > 0$. $I_\pi = \{d : d \cdot (\pi_1 - \pi_2) > 0,\ d \cdot (\pi_2 - \pi_3) > 0\}$ is the intersection of two open half-circles; it is non-empty because $\pi_1 - \pi_2$ and $\pi_2 - \pi_3$ are not opposite (T is not aligned), and shorter than a half-circle because they are not equal. A direction outside the six arcs is orthogonal to some $z_j - z_k$, and there are six such directions. (ii) $\mathbb{E}|x - Y|^2 = |x - \mathbb{E}Y|^2 + \operatorname{Var} Y$, and the difference of two such expressions is affine in x with linear part $2\,x \cdot n_i(A)$. (iii) Along the ray, $G_i = 2t\,d \cdot n_i + O(1)$, and a disc of fixed radius around $\omega + td$ is at distance of order t from the boundary of the cone S_π and from the line $\{G_i = \kappa\}$. (iv) Sector masses depend only on the sector of each disc. □

Proof of Proposition 2. Let $m \in \Gamma_T$ and $(\alpha_a, \alpha_b, \alpha_c) \in [0, m_a] \times [0, m_b] \times [0, m_c]$. Put $\lambda_{cab} = \alpha_c$, $\lambda_{cba} = m_c - \alpha_c$, $\lambda_{abc} = \alpha_a$, $\lambda_{acb} = m_a - \alpha_a$, $\lambda_{bca} = \alpha_b$, $\lambda_{bac} = m_b - \alpha_b$, and realize these masses by Lemma S19(iv). The first-choice masses are $m \in \Gamma_T$, so T is offered, and $s_{ab} = m_a + \alpha_c$, $s_{bc} = m_b + \alpha_a$, $s_{ca} = m_c + \alpha_b$: every point of $B(m)$ is produced, and arbitration is free. In the two cases, Corollary 1(i)–(ii) gives the nullity of every valid rule that is non-imposed on the shares of T. If every share of a pair of T produced at some offer is also produced with T offered, then $D_{ab} = \pi_{ab}(V_T)$ for each pair, the boxes $B(m)$ with $\max m \le \frac{1}{2}$ are products through h whose factors cover the share ranges, and T is a star triple. □

Plurality without noise. Here $Y_A = z_{W(A)}$ with $W(A)$ the plurality winner, and $G_j(x; A) = |x - z_{W(A \setminus j)}|^2 - |x - z_{W(A)}|^2$. Hence:

Lemma S20 (Useless parties die)**.** *If* $W(A \setminus j) = W(A)$ *then* $S_j(A) = 0$*. If* $W(A \setminus j) = k \neq W(A)$ *then* $S_j(A) = s^{\kappa}_{W(A)\,k}$*.*

Proof of Proposition 3. (i) In a pair, the loser j has $W(A \setminus j) = W(A)$, so $S_j = 0 < \varepsilon$ by Lemma S20. With more births, this holds whatever the outsiders do, since they only add conditions.

(ii) Let $a = W(T)$. Removing b leaves the duel $\{a, c\}$. If a wins it, b is useless; if c wins it, $S_b = s^{\kappa}_{ac}$. Symmetrically c is useless unless b beats a, and then $S_c = s^{\kappa}_{ab}$. Removing a leaves $\{b, c\}$, won by some k, so $S_a = s^{\kappa}_{ak}$, one of the two masses above. There are no outsiders. So T is stable if and only if b and c both beat a and $\min(s^{\kappa}_{ab}, s^{\kappa}_{ac}) \geq \varepsilon$.

Validity: on an offer, the plurality winner loses both duels; a tournament on three parties with a Condorcet loser has no cycle, and there are no exact ties. In a cyclic triple every party wins one duel, so no plurality winner loses both, and the society has no offer. Majority reads only s_{xy}, so it is independent and anonymous.

Non-imposition: let $\varepsilon' \in (\max(\varepsilon, \frac{1}{3}), \frac{1}{2})$ and $\eta > 0$ small. Give a first-choice mass ε', b mass $(1-\varepsilon')/2 + \eta$ and c mass $(1-\varepsilon')/2 - \eta$, all members of b and c ranking a last: $\lambda_{bca} = (1-\varepsilon')/2 + \eta$, $\lambda_{cba} = (1-\varepsilon')/2 - \eta$, $\lambda_{bac} = \lambda_{cab} = 0$, $\lambda_{abc} = \theta$, $\lambda_{acb} = \varepsilon' - \theta$. For η small, a wins plurality since $\varepsilon' > \frac{1}{3}$, and $s_{ab} = s_{ac} = \varepsilon' \in [\varepsilon, \frac{1}{2})$, so b and c beat a. The duel $\{b, c\}$ has $s_{bc} = (1-\varepsilon')/2 + \eta + \theta$, which crosses $\frac{1}{2}$ as θ ranges over $(0, \varepsilon')$; the first-choice masses in each contest are distinct except at the one value of θ where this duel is tied. Place the masses far out (Lemma S19(iv)): loss differences are then of order t, so $s^{\kappa} = s$ and the triple is stable. Permuting the labels gives every strict order whose last party is the plurality winner, and each ordered pair is ranked strictly both ways on intervals of shares.

(iii) If no party survives there is no government, which every citizen ranks below every platform by more than any cost; so $S_j(\{j\}) = 1 \geq \varepsilon$. For $k \neq j$, in $\{j, k\}$ the support of k is 0 if j beats k (Lemma S20) and $s^{0}_{kj} = s_{kj} > \frac{1}{2} > \varepsilon$ otherwise. So $\{j\}$ is stable if and only if j beats both other parties. In a stable triple, the plurality winner loses both duels, and the winner of the third duel beats both other parties.

(iv) Put W at the origin and b, c, d on the unit circle at angles 90°, 210°, 330°; write $P(r, \theta)$ for the point at distance r and angle θ. The society is uniform on small discs around nine points:

- members of W, mass 0.28: one group near each of $P(0.1, \theta)$ for $\theta = 120°, 60°, 180°, 240°, 0°, 300°$, ranking W first and b, c, d in the orders *bcd*, *bdc*, *cbd*, *cdb*, *dbc*, *dcb* respectively; the two groups whose order starts with b have mass 0.05, the four others 0.045;
- members of b, mass 0.245, near $P(3, 140°)$, ranking $b \succ c \succ W \succ d$;
- members of c, mass 0.24, near $P(3, 260°)$, ranking $c \succ d \succ W \succ b$;
- members of d, mass 0.235, near $P(3, 20°)$, ranking $d \succ b \succ W \succ c$.

The rankings follow from the distances; at $P(3, 140°)$, for instance, the squared distances to b, c, W, d are $10 - 6\cos 50° = 6.14$, $10 - 6\cos 70° = 7.95$, 9 and $10 - 6\cos 190° = 15.91$. Every loss difference that decides a support below is at least 0.8 at the centres of the discs (the smallest, for members of W, is $1 - 0.2\cos(\cdot) \geq 0.8$), so for small discs the conclusions hold for every $\kappa < 0.8$; scaling the figure scales this bound.

W wins plurality (0.28 against 0.245, 0.24, 0.235). Removing b sends its members to c, which wins with 0.485; the supporters of b are those who prefer W to c, the members of W and of d, mass 0.515. Removing c makes d win with 0.475, and c is supported by the members of W and b (0.525); removing d makes b win with 0.48, and d is supported by the members of W and c (0.52). Removing W gives b $0.245 + 0.10$, c $0.24 + 0.09$ and d $0.235 + 0.09$, so b wins, and W is supported by the members of W and c (0.52). There are no outsiders, so

the offer is stable for every $\varepsilon \le 0.515$. The shares are $s_{bc} = 0.245 + 0.235 + 0.145 = 0.625$, $s_{cd} = 0.24 + 0.245 + 0.14 = 0.625$ and $s_{db} = 0.235 + 0.24 + 0.135 = 0.61$ (own members, the members who rank the winner second, and the members of W with that order), so $b \succ c \succ d \succ b$. In every contest among the four parties the first-choice masses are distinct, and no duel is tied.

Runoff with two or three births: in a pair the loser is useless. In a triple, the party with the smallest first-choice mass is eliminated and the outcome is the winner of the duel between the two others; without it, the same two parties meet in the same duel, so it is useless. If a party wins a first-round majority, it also wins the duel, and the argument is unchanged. In a stable triple every party has support at least $\varepsilon > 0$, so no triple is stable. □

Remark S8 (Runoff with four births). With four births, removing a party eliminated in the first round redistributes its votes and can change who reaches the duel. Let the births be $p = (2.5, 1.1)$, $q = (1.1, 1.7)$, $r = (0.5, -1)$, $s = (-1.2, -1.2)$, and let the society be uniform on four discs of radius 0.01: mass 0.30 around $(6, 1.4)$, ranking $pqrs$; 0.26 around $(-0.15, 0.9)$, $qrsp$; 0.23 around $(3.1, -6)$, $rspq$; 0.21 around $(-0.4, -6)$, $srpq$. With all four, p and q reach the duel and p wins (0.74). Without p, q wins, and p is supported by the 0.74 who prefer p to q. Without q, r reaches the duel and beats p; without r, s does; without s, r does; in each case the party removed is supported by the 0.30 who prefer p to the new winner. No party is useless, every loss difference involved is at least 2.69, and the offer of all four parties is stable for every $\varepsilon \le 0.30$ and $\kappa < 2.69$.

Positive weights. For the three rules, write $\mathbb{E}Y_T = \sum_{j \in T} \beta_j z_j$ with $\beta_j \ge 0$ and $\sum_j \beta_j = 1$, and, for the pair $T \setminus i = \{k, l\}$, $\mathbb{E}Y_{T \setminus i} = q^i_k z_k + q^i_l z_l$ with $q^i_k + q^i_l = 1$. Under plurality or runoff, β_j is the probability that j wins; under the compromise, $\beta_j = \mathbb{E}w_j$. Write H_{kl} for the set of the three rankings in which k is above l; its mass is s_{kl}.

Lemma S21 (Half-circles)**.** *Let $n \neq 0$, $\Sigma = \{d : d \cdot n > 0\}$, and $A(n)$ the set of rankings π with $I_\pi \cap \Sigma \neq \emptyset$. Then $A(n) \supseteq H_{kl}$ for some ordered pair (k, l). If n is not a positive multiple of any $z_k - z_l$, then $A(n) \supseteq H_{kl} \cup H_{k'l'}$ for two different ordered pairs.*

Proof. By Lemma S19(i), the six separating directions are orthogonal to the three edges of T, so they come in antipodal pairs; in circular order $s_1, \dots, s_6$, the antipode of s_j is s_{j+3}, and the arcs between s_j and s_{j+3} on either side are three consecutive arcs I_π. Such a half-circle is $\{d : d \cdot (z_k - z_l) > 0\}$ for the edge orthogonal to s_j and the right orientation, minus two separating directions, and its three arcs are those of the rankings in H_{kl}. If n is a positive multiple of $z_k - z_l$, then $A(n) = H_{kl}$. Otherwise the two endpoints of Σ are not separating directions; they lie inside two antipodal arcs, and Σ meets these two arcs and contains the two arcs between them: four consecutive arcs, which contain two different sets of three consecutive arcs. □

Proof of Theorem 4. Step 1: weights. Under plurality with noise, every party has the largest shocked mass with positive probability. Under runoff with noise, every party reaches the duel and wins it with positive probability, the duel being noisy too. Under the compromise, $\beta_j = \mathbb{E}w_j > 0$ when $m_j(T) > 0$, and with noise also when $m_j(T) = 0$, since the shocked mass is positive with positive probability. When $\beta_i > 0$, the coefficients of $n_i = \bar{Y}(T) - \bar{Y}(T \setminus i) = \beta_i z_i + \sum_{j \neq i} (\beta_j - q^i_j) z_j$ sum to zero, so $n_i = \beta_i (z_i - p_i)$ with p_i on the line through the two other platforms, and $n_i \neq 0$.

Step 2: the arm. For $w \in [0, \frac{1}{2}]$ let

$$\lambda_{abc} = 0, \quad \lambda_{acb} = w, \quad \lambda_{bac} = \tfrac{1}{2} - w, \quad \lambda_{bca} = w, \quad \lambda_{cab} = 0, \quad \lambda_{cba} = \tfrac{1}{2} - w.$$

Then $s_{ab} = w$, $s_{bc} = s_{ca} = \frac{1}{2}$, and $m_a(T) = w$, $m_b(T) = \frac{1}{2}$, $m_c(T) = \frac{1}{2} - w$. Every ordered pair other than (a, b) has $s_{kl} \geq \frac{1}{2}$. Suppose $\beta_j > 0$ for all j (always, under the noisy rules; for $w \in (0, \frac{1}{2})$ under the deterministic compromise). I claim that no n_i is a positive multiple of $a - b$.

- $i = c$: $c - p_c$ joins the line (ab) to c, which is off that line, so it is not parallel to $a - b$.
- $i = b$: if $b - p_b = \tau(a - b)$ with $\tau > 0$, then $p_b = (1 + \tau)b - \tau a$ lies on the line (ab) and on the line (ac), so $p_b = a$ and $\tau = -1$, a contradiction.
- $i = a$: $a - p_a$ is parallel to $a - b$ only if $p_a = b$, that is, only if the coefficient of z_c in n_a vanishes: $\beta_c = q_c^a$. The members of a all rank c second, so in the pair $\{b, c\}$ the first-choice masses are $\frac{1}{2}$ and $(\frac{1}{2} - w) + w = \frac{1}{2}$, and the duel is tied; each rule treats the two parties alike, and $q_c^a = \frac{1}{2}$. In T, $\beta_c < \frac{1}{2}$. Under plurality with noise, exchanging the shocks of b and c, which have the same law, shows $\beta_c \leq \beta_b$, with equality only if $w = 0$; so $\beta_c < \frac{1}{2}$ when $w > 0$, and $\beta_c = \frac{1-\beta_a}{2} < \frac{1}{2}$ when $w = 0$. Under the compromise, the same exchange applies, since w_c increases with the mass of c and decreases with the mass of b. Under runoff, the duels $\{b, c\}$ and $\{c, a\}$ are tied, so $\beta_c = \frac{1}{2}(1 - P(\text{the duel is } \{a, b\})) < \frac{1}{2}$, as c has the smallest shocked mass with positive probability.

By Lemma S21, each $A(n_i)$ contains some H_{kl} with $(k, l) \neq (a, b)$, hence carries mass at least $\frac{1}{2}$. Split each λ_π into three equal pieces; place the piece of index i in a direction of $I_\pi \cap \Sigma_i$ when this set is non-empty, anywhere in I_π otherwise, and all pieces far out (Lemma S19(iii)–(iv)). Then $S_i(T) \geq \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \geq \varepsilon$ for each i, and T is stable, there being no outsiders. So $(w, \frac{1}{2}, \frac{1}{2}) \in V_T$.

Step 3: the centre and the crosses. With $\lambda_\pi = \frac{1}{6}$ for all π, every H_{kl} has mass $\frac{1}{2}$, every party has first-choice mass $\frac{1}{3}$, and the same placement gives supports at least $\frac{1}{6}$; so $h \in V_T$. Exchanging the labels a and b in Step 2 gives $(1-w, \frac{1}{2}, \frac{1}{2})$, and cyclic relabelings give the arms of the two other coordinates; the rules treat the parties alike. Under the noisy rules the arms cover $[0, 1] \supseteq D_{ab}$. Under the deterministic compromise they cover $(0, 1)$, together with h; and $0 \notin D_{ab}$: if $s_{ab} = 0$ at an offer containing a and b, then a has no first-choice mass there, it does not move the compromise, $G_a \equiv 0$ and $S_a = 0 < \varepsilon$; likewise $1 \notin D_{ab}$. So V_T contains crosses through h whose factors cover the share ranges, T is a star triple, and Theorem 1 applies, since every offered pair lies in T. Majority rule is anonymous, independent, non-null, and non-imposed here, since each pair is produced on both sides of one half; so it is not valid. □

Bounded societies. Take ω as the origin and let R_T be the circumradius of T. By Lemma S19(ii), $G_i(x; A) = 2\,x \cdot n_i(A) + c_i(A)$.

Lemma S22 (Lines of support)**.** *(i) Under plurality and runoff, with or without noise, $c_i(A) = 0$: with $\kappa = 0$ the supporters of i form the open half-plane $\{x : x \cdot n_i(A) > 0\}$, bounded by a line through ω.*

(ii) Under the deterministic compromise, $G_i(x; A) = 2\,n_i(A) \cdot (x - o_i)$ with $o_i = \frac{1}{2}(\bar{Y}(A) + \bar{Y}(A \setminus i))$, a point of the triangle; so $\{G_i > 0\}$ contains $\{x : x \cdot n_i > R_T\,|n_i|\}$.

Proof. (i) Y_A and $Y_{A\setminus i}$ are parties, and every party z has $|z| = R_T$, so $\mathbb{E}|x - Y|^2 = |x|^2 - 2\,x \cdot \mathbb{E}Y + R_T^2$ for both, and the difference is $2\,x \cdot n_i$. (ii) Without noise, $G_i = |x - \bar{Y}(A \setminus i)|^2 - |x - \bar{Y}(A)|^2$, which is $2\,n_i \cdot (x - o_i)$; and $|o_i| \leq R_T$ since the triangle lies in the circumscribed disc. □

Two elementary facts serve below. Let d be a unit vector and $\hat{u} = u/|u|$.

(C) If $n = s\,u + t\,v$ with $s, t \geq 0$ not both zero, and $d \cdot u > 0$, $d \cdot v > 0$, then $d \cdot n \geq (s|u| + t|v|) \min(d \cdot \hat{u}, d \cdot \hat{v}) \geq |n| \min(d \cdot \hat{u}, d \cdot \hat{v})$.

(R) If $n = s(u + r\,v)$ with $s > 0$ and $0 \leq r \leq \bar{r}$, and $d \cdot v < 0$, then $d \cdot n \geq s\,(d \cdot u + \bar{r}\,d \cdot v)$ and $|n| \leq s(|u| + \bar{r}|v|)$.

Proof of Proposition 4. Restricting the domain to societies carried by Δ can only shrink the share ranges D_{ab}; so it is enough to produce, with societies carried by Δ, the centre and the arms used in the proof of Theorem 4. Sectors are open cones with apex ω, so $t\,d \in S_\pi$ for every $t > 0$ when $d \in I_\pi$.

Plurality and runoff with noise, $\kappa = 0$. By Lemma S22(i), $\{G_i > 0\}$ is the open half-plane $\{x \cdot n_i > 0\}$, a cone with apex ω. In the placement of Steps 2 and 3 of that proof, a piece in a direction $d \in I_\pi$ with $d \cdot n_i > 0$ supports i at every distance $t > 0$. Put every piece on a small disc around $t\,d$ with t less than half the radius of Δ. The supports are those of the proof, and the conclusion follows as there.

The noisy rules. Under the noisy rules, the weights β_j and q^i_j are Gaussian integrals of functions of the first-choice masses and of the shares that are bounded and continuous outside a null set, so they, n_i and c_i are continuous in w on the closed arm $w \in [0, \frac{1}{2}]$. Step 1 of that proof gives $n_i \neq 0$, and Step 2 shows that n_i is never a positive multiple of $a - b$. The centre is the arm point $w = \frac{1}{2}$, and the six arms are obtained by relabelling; all the constants below are minima over the six arms. For each i let

$$M_i(w) = \max_{(k,l) \neq (a,b)} \; \min_{\pi \in H_{kl}} \; \max_{d \in \overline{I_\pi}} \; d \cdot \hat{n}_i(w).$$

By Lemma S21, some H_{kl} with $(k, l) \neq (a, b)$ lies in $A(n_i)$, so $M_i(w) > 0$; and M_i is continuous. On the compact arm, $M_i \geq M_0 > 0$, $|n_i| \geq \nu_0 > 0$ and $|c_i| \leq \bar{c}$. Given w, take the (k, l) that attains $M_i(w)$ and, for each $\pi \in H_{kl}$, a direction $d \in I_\pi$ with $d \cdot \hat{n}_i > M_0/2$. A piece placed around $t\,d$ then has $G_i \geq t\,M_0\nu_0 - \bar{c}$, which exceeds κ as soon as $t > (\kappa + \bar{c})/(M_0\nu_0)$. With these directions the three-way split of Step 2 gives supports at least $\frac{1}{6}$. A disc of radius $2(\kappa + \bar{c})/(M_0\nu_0)$ leaves room for the small discs; with $\nu = M_0\nu_0/2$ and $R_0 = 2\bar{c}/(M_0\nu_0)$ this is $R_0 + \kappa/\nu$, as claimed, for every $\kappa \geq 0$. Under plurality and runoff, $\bar{c} = 0$, so $R_0 = 0$.

The compromise without noise, $\kappa = 0$. Here the arm is $w \in (0, \frac{1}{2})$, and the first-choice masses are $m(T) = (w, \frac{1}{2}, \frac{1}{2} - w)$, $m(\{b, c\}) = (\frac{1}{2}, \frac{1}{2})$, $m(\{a, c\}) = (\frac{1}{2}, \frac{1}{2})$ and $m(\{a, b\}) = (w, 1 - w)$. So $\beta_j = m_j^\gamma/\Sigma$ with $\Sigma = w^\gamma + 2^{-\gamma} + (\frac{1}{2} - w)^\gamma$, $q^a = q^b = (\frac{1}{2}, \frac{1}{2})$ and $q^c_a = w^\gamma/(w^\gamma + (1 - w)^\gamma)$. For $\gamma \geq 1$, $s^\gamma + t^\gamma \leq (s + t)^\gamma$, so $w^\gamma + (\frac{1}{2} - w)^\gamma \leq 2^{-\gamma}$, hence $\beta_b \geq \frac{1}{2}$ and $\beta_a, \beta_c \leq \frac{1}{2}$; and $2^{-\gamma} + (\frac{1}{2} - w)^\gamma \leq (1 - w)^\gamma$, hence $\beta_a \geq q^c_a$. Since the coefficients of each n_i sum to zero,

$$n_a = \beta_a(a - c) + (\beta_b - \tfrac{1}{2})(b - c), \qquad n_b = (\tfrac{1}{2} - \beta_a)(b - a) + (\tfrac{1}{2} - \beta_c)(b - c),$$
$$n_c = \beta_c(c - b) + (\beta_a - q^c_a)(a - b),$$

all coefficients non-negative. The ratios are bounded where they will be used: for $w \leq \frac{1}{3}$, $(\beta_a - q^c_a)/\beta_c \leq \beta_a/\beta_c = (w/(\frac{1}{2} - w))^\gamma \leq 2^\gamma$; for $w \geq \frac{1}{3}$, $(\beta_b - \frac{1}{2})/\beta_a = (2^{-\gamma} - w^\gamma - (\frac{1}{2} - w)^\gamma)/(2w^\gamma) \leq (\frac{3}{2})^\gamma$.

Fix four directions, depending only on T and γ: $d_1 \in I_{bac}$ and $d_4 \in I_{bca}$ arbitrary; $d_2 \in I_{cba}$ with $d_2 \cdot (c - b) > 2^{\gamma+1}\,d_2 \cdot (b - a)$, which exists because at the direction that separates *cba* from *cab*, $d \cdot (b - a) = 0 < d \cdot (c - b)$; and $d_3 \in I_{acb}$ with $d_3 \cdot (a - c) > 2(\frac{3}{2})^\gamma\,d_3 \cdot (c - b)$, which exists for the same reason at the direction that separates *acb* from *abc*. Then, for every w in the range indicated:

- d_1 ($b \succ a \succ c$) has $d_1 \cdot (a - c), d_1 \cdot (b - c), d_1 \cdot (b - a) > 0$, so by (C) $d_1 \cdot \hat{n}_a$ and $d_1 \cdot \hat{n}_b$ are bounded below by a positive constant; likewise $d_4 \cdot \hat{n}_b$ (b first);

- for $w \le \frac{1}{3}$, (R) with $u = c - b$, $v = a - b$, $\bar{r} = 2^\gamma$ bounds $d_2 \cdot \hat{n}_c$ below by $\big(d_2 \cdot (c-b) - 2^\gamma d_2 \cdot (b-a)\big)/\big(|c-b| + 2^\gamma |a-b|\big) > 0$;
- for $w \ge \frac{1}{3}$, (R) with $u = a - c$, $v = b - c$, $\bar{r} = (\frac{3}{2})^\gamma$ bounds $d_3 \cdot \hat{n}_a$ below, and (C) bounds $d_3 \cdot \hat{n}_c$ below, since $a \succ c \succ b$ gives $d_3 \cdot (c-b), d_3 \cdot (a-b) > 0$.

Let $\eta > 0$ be the smallest of these bounds and $t = 2R_T/\eta$. By Lemma S22(ii), a piece on a small disc around $t\,d$ supports i whenever $d \cdot \hat{n}_i \ge \eta$. For $w \le \frac{1}{3}$, put the mass $\frac{1}{2} - w$ of bac around $t\,d_1$, the mass $\frac{1}{2} - w$ of cba around $t\,d_2$, and the two other masses anywhere in their sectors at distance t: then $S_a, S_b, S_c \ge \frac{1}{2} - w \ge \frac{1}{6}$. For $w \ge \frac{1}{3}$, put the mass w of acb around $t\,d_3$, the mass w of bca around $t\,d_4$, and the two other masses anywhere in their sectors at distance t: then $S_a, S_c \ge w$ and $S_b \ge w$, all at least $\frac{1}{3}$. At the centre, $\lambda_\pi = \frac{1}{6}$ gives $\beta_j = \frac{1}{3}$ and $q_j^i = \frac{1}{2}$, so $n_i = \frac{1}{6}\big((i-j) + (i-k)\big)$. Put the mass of each sector around $t\,d_\pi$ for a fixed $d_\pi \in I_\pi$; if i is first in π, then $d_\pi \cdot (i-j)$ and $d_\pi \cdot (i-k)$ are positive, and (C) bounds $d_\pi \cdot \hat{n}_i$ below. Taking η also below these six bounds, $S_i \ge \frac{1}{3}$ for each i. Relabelling gives the other arms, with η the smallest bound over the six arms, and $0, 1 \notin D_{ab}$ as in Step 3. So the conclusion holds on every disc of radius $R_0 = 3R_T/\eta$, which leaves room for the small discs. □

Remark S9 (Distances and costs). The constant η of the last step is crude: in the numerical checks (file `check_bounded_kappa0.py` of the project, six arms, pieces on discs around $2R_T/\eta$), the radius $R_0 = 3R_T/\eta$ lies between 40 and 600 circumradii for $\gamma \in \{1, 1.5, 2, 3, 5\}$, and it grows with γ and as the triangle flattens; a linear program over placements (file `probe_bounded_society.py`, three triangles) finds supports of at least 0.45 within 1.5 circumradii for $\gamma = 1$. Under the noisy compromise, the three-way split works within 1.3 to 8 circumradii for $\gamma \in \{1, 3\}$ and $\sigma \in \{0.05, 0.1\}$. With $\kappa > 0$ the radius needed grows like $\kappa/|n_i|$, and $|n_i| \le \beta_i\,|z_i - p_i|$ is small when β_i is. Under plurality with $\sigma = 0.05$, a party with no members facing two parties with half of the citizens each wins with probability of order 10^{-17}, and the radius needed is of order 10^{14} times the size of the triangle. In numerical checks with $\kappa = 0.02$ on random triangles (file `check_arms_positive.py`), every arm point and the centre are produced within $2 \cdot 10^4$ times the size of the triangle for plurality and runoff with $\sigma = 0.1$, and for the compromise with $\gamma \in \{1, 3\}$ and $\sigma \in \{0, 0.05\}$ at the arm points tested. Under the deterministic compromise, the distance grows without limit at the ends of the arms, like $w^{-\gamma}$, and with $\kappa > 0$ the bounded version is open. Proposition 3 needs no far placement when $\kappa = 0$; for $\kappa > 0$, its non-imposition construction uses it to remove the band of nearly indifferent citizens.

S3 Parties at the mean of their members: selected offers

This section proves the second half of Proposition 6: in Example 1, under the optimal selection, and under any selection that places the parties on the clusters when the citizens form k tight, well-separated clusters, every offered pair belongs to a star triple. I use the notation of Appendix B: $T = \{a, b, c\}$, ω its circumcentre, $\mathrm{Eq}(\mu)$ the set of equilibrium offers of μ, $f(\pi)$ the share vector of a citizen with strict order π on T, $\mathcal{P}^\circ$ the open polytope of Lemma 6(c), and r_t the open ray that splits the cell of t relative to T into its two sectors (Lemma 6(b)). Distributions charge no line and, under the optimal selection, have a finite second moment. For a finite set Q of points, the *distortion* of μ is $\Delta_\mu(Q) = \int \min_{q \in Q} |x - q|^2 \, d\mu(x)$.

Lemma S23 (Clusters)**.** *Let $z_1, \dots, z_k$ be distinct points and $L = \min_{i \ne j} |z_i - z_j|$. Let $\mu = \mu_1 + \cdots + \mu_k$ be a probability measure, where μ_i has mass $w_i > 0$, is carried by the closed ball $\bar{B}(z_i, \rho)$ and has barycentre z_i. Let $\eta = \min_i w_i$. If*

$$\rho < \frac{\sqrt{\eta}\, L}{3(1 + \sqrt{\eta})},$$

then $Z = \{z_1, \dots, z_k\}$ is the unique set of at most k points that minimizes Δ_μ, and every point of $\bar{B}(z_i, \rho)$ is strictly closer to z_i than to any other point of Z.

Proof. For $\rho \geq 0$ the hypothesis is equivalent to $\rho(1+\sqrt{\eta}) < \sqrt{\eta}\, L/3$, that is, $\rho < \sqrt{\eta}\,(L/3-\rho)$, which forces $\rho < L/3$; squaring, it is equivalent to

$$\rho^2 < \eta\Big(\frac{L}{3} - \rho\Big)^2 \quad \text{with} \quad \rho < \frac{L}{3}.$$

Since $\eta \leq 1$, $\sqrt{\eta}/(1+\sqrt{\eta}) \leq 1/2$, and the hypothesis also gives $\rho < L/6$.

An upper bound. Each point of the cluster i is within ρ of z_i, and the total mass is 1, so $\Delta_\mu(Z) \leq \sum_i \int |x - z_i|^2 \, d\mu_i \leq \rho^2$.

Let Q be a set of at most k points with $\Delta_\mu(Q) \leq \Delta_\mu(Z)$.

Step 1: Q has one point near each z_i. Suppose Q misses the open ball $B(z_i, L/3)$. Then every point of the cluster i is at distance at least $L/3 - \rho > 0$ from Q, and

$$\Delta_\mu(Q) \;\geq\; w_i\Big(\frac{L}{3} - \rho\Big)^2 \;\geq\; \eta\Big(\frac{L}{3} - \rho\Big)^2 \;>\; \rho^2 \;\geq\; \Delta_\mu(Z),$$

a contradiction. The k balls $B(z_i, L/3)$ are disjoint, since their centres are at least L apart. As Q has at most k points and meets each ball, it has exactly one point q_i in each ball and no other point.

Step 2: each cluster goes entirely to its own point. Let x be in the cluster i and $j \neq i$. Then $|x - q_i| < \rho + L/3$ and $|x - q_j| \geq |z_i - z_j| - \rho - L/3 \geq 2L/3 - \rho$. Since $\rho < L/6$, $2L/3 - \rho > L/3 + \rho$, so x is strictly closer to q_i than to q_j.

Step 3: conclusion. By Step 2, $\Delta_\mu(Q) = \sum_i \int |x - q_i|^2 \, d\mu_i$. Since z_i is the barycentre of μ_i, $\int |x - q|^2 \, d\mu_i = \int |x - z_i|^2 \, d\mu_i + w_i |q - z_i|^2$ for every q. Step 2 applied to $Q = Z$ gives $\Delta_\mu(Z) = \sum_i \int |x - z_i|^2 \, d\mu_i$. Hence

$$\Delta_\mu(Q) = \Delta_\mu(Z) + \sum_i w_i |q_i - z_i|^2.$$

Since $\Delta_\mu(Q) \leq \Delta_\mu(Z)$ and $w_i > 0$, $q_i = z_i$ for all i, and $Q = Z$. Step 2 with $Q = Z$, which holds for every point of $\bar{B}(z_i, \rho)$, gives the last claim. □

For instance, three clusters of mass $1/3$ at the vertices of an equilateral triangle of side 1 give $\eta = 1/3$, $L = 1$, and the threshold $\sqrt{1/3}/(3(1+\sqrt{1/3})) \approx 0.122$. With $\rho = 0.1$, the offer of the vertices costs at most 0.01, while an offer that misses a cluster costs at least $\frac{1}{3}(\frac{1}{3} - 0.1)^2 \approx 0.018$.

As in Appendix S1.1, $\mathrm{Opt}(\mu)$ is the set of sets of at most k points that minimize Δ_μ, and the *optimal selection* picks some $\sigma(\mu) \in \mathrm{Opt}(\mu)$ whenever this set is non-empty, with any tie-breaking. Its domain $\mathcal{D}_\sigma = \{(x, \sigma(\mu_x))\}$ satisfies Assumption 1, so Lemma 1 and Theorem 1 apply, with D_{ab} and V_T computed on $\mathcal{D}_\sigma$.

Lemma S24 (Optimal offers are equilibria)**.** *Let μ charge no line and have a finite second moment, and let $Q \in \mathrm{Opt}(\mu)$. Then Q has exactly k points, each cell of Q has positive mass, and each point of Q is the mean of its cell. Hence* $\mathrm{Opt}(\mu) \subset \mathrm{Eq}(\mu)$.

Proof. Cells are non-empty. Suppose that $|Q| < k$, or that some cell W_q has mass zero. In the second case, removing q does not change the distortion: only the points of the closed cell of q change their nearest platform, and this closed cell has mass zero, since its boundary is made of pieces of lines, which μ does not charge. In both cases there is a set Q' of at most $k-1$ points with $\Delta_\mu(Q') = \Delta_\mu(Q) < \infty$. The support of μ is infinite, since μ charges no line. Let y be a point of the support outside Q' and $r = \mathrm{dist}(y, Q')/2 > 0$. The ball $B(y, r)$

has positive mass, and each x in it satisfies $|x-y| < r < |x-q'|$ for every $q' \in Q'$. So $\Delta_\mu(Q' \cup \{y\}) < \Delta_\mu(Q)$, against optimality.

Means. Replace each q by the mean g_q of its cell while keeping the cells. This lowers the distortion by $\sum_q \mu(W_q)|q-g_q|^2$. Reassigning each citizen to the nearest new point lowers it further, weakly. By optimality, $q = g_q$ for every q. □

Lemma S25 (Richness under the optimal selection)**.** *Let $k \geq 3$, let $a \neq b$, and let c be such that $T = \{a, b, c\}$ is equilateral. For every $(p, q, r) \in \mathcal{P}^\circ$ there is a distribution μ with compact support whose* unique *optimal set of k points is an offer $O \supseteq T$, and whose shares are $(s_{ab}, s_{bc}, s_{ca}) = (p, q, r)$. The construction can be made with every cluster in a ball of radius ρ, for every small enough $\rho > 0$, with L and η as in Lemma S23 independent of ρ.*

Proof. Why equilateral. In an equilateral triangle each vertex lies on the bisector of the opposite side, so $t \in r_t$ for $t \in T$. A small cluster centred at t can then have members on both sides of r_t, in any proportions, and still have mean t.

Far parties, if $k > 3$. Choose $k-3$ distinct unit vectors e_j such that each $\omega + e_j$ lies in a sector of T, let π_j be the order of that sector, and let u be the mean of the $f(\pi_j)$. Each far party will carry a mass $\eta_0 > 0$; let $\delta = (k-3)\eta_0$. Since (p, q, r) is interior to $\mathcal{P}$, $(p', q', r') = \big((p, q, r) - \delta u\big)/(1-\delta) \in \mathcal{P}^\circ$ for η_0 small; fix such an η_0. By Lemma 8(i) and (iii), applied with $v_j = e_j$ and any $R > 0$, there is τ such that the points $d_j = \omega + \tau e_j$ are distinct, lie off T, and $\bar{B}(d_j, 1)$ lies in the sector of π_j. For $k = 3$, $\delta = 0$ and there is no far party.

Sector masses. By Lemma 6(c), there are six strictly positive masses w_π of sum $1-\delta$ whose shares are $(1-\delta)(p', q', r')$. For $t \in T$, let π_1, π_2 be the two orders that rank t first, and $m_1 = w_{\pi_1}$, $m_2 = w_{\pi_2}$.

Cluster of t. Let e be a unit vector orthogonal to the line (ωt), pointing towards the sector of π_1. Let $\rho_1 = \frac{\rho}{2} \cdot \frac{m_2}{m_1+m_2}$ and $\rho_2 = \frac{\rho}{2} \cdot \frac{m_1}{m_1+m_2}$. Put the mass m_1 uniformly on the disc of centre $t + \rho_1 e$ and radius $\min(\rho_1, \rho_2)/2$, and the mass m_2 uniformly on the disc of centre $t - \rho_2 e$ with the same radius. The barycentre is $t + \frac{m_1\rho_1 - m_2\rho_2}{m_1+m_2}\, e = t$, and the cluster lies in $\bar{B}(t, \rho)$. The first disc is strictly on one side of the line (ωt) and the second strictly on the other side. For ρ small, $\bar{B}(t, \rho)$ lies in the cell of t relative to T and meets the line (ωt) only on r_t, so the first disc lies in the sector of π_1 and the second in the sector of π_2.

Clusters of the far parties. Put the mass η_0 uniformly on $\bar{B}(d_j, \rho)$, with $\rho \leq 1$; its barycentre is d_j, and it lies in the sector of π_j.

Conclusion. Let $Z = T \cup \{d_1, \ldots, d_{k-3}\}$, L the smallest distance between two points of Z, and η the smallest cluster mass, which is positive. Neither depends on ρ. Choose $\rho < \sqrt{\eta}\, L/(3(1+\sqrt{\eta}))$, and small enough for the conditions of position above. By Lemma S23, Z is the unique optimal set, and its cells are the clusters. The sectors carry the masses w_π, the far discs carry the order of their sectors, and the shares are $(1-\delta)(p', q', r') + \delta u = (p, q, r)$. □

The construction of Lemma 9 does not serve here. On a triangle that is not equilateral, t is at a fixed positive distance from r_t, the two discs of a cell move far from t when one sector has a small mass, and the optimal set may group the discs differently.

Definition S4 (Recognizing clusters)**.** A selection σ, defined on a set $\mathrm{dom}(\sigma)$ of distributions with $\sigma(\mu) \in \mathrm{Eq}(\mu)$, *recognizes obvious clusters* if there is a function $\bar{\rho}(\eta, L)$ with $0 < \bar{\rho}(\eta, L) \leq L/2$ such that the following holds. If $z_1, \ldots, z_k$ are distinct and pairwise at distance at least L, if $\mu = \sum_i \mu_i$ with μ_i carried by $\bar{B}(z_i, \rho)$, of mass at least η and barycentre z_i, where $\rho < \bar{\rho}(\eta, L)$, and if $\{z_1, \ldots, z_k\} \in \mathrm{Eq}(\mu)$, then $\mu \in \mathrm{dom}(\sigma)$ and $\sigma(\mu) = \{z_1, \ldots, z_k\}$.

The condition $\bar{\rho} \leq L/2$ makes the definition consistent: the balls are then disjoint, the decomposition $\mu = \sum_i \mu_i$ is unique, and the barycentres are determined by μ. The optimal selection recognizes obvious clusters, with $\bar{\rho}(\eta, L) = \sqrt{\eta}\, L/(3(1+\sqrt{\eta})) \leq L/6$. Indeed, this

threshold increases in η and in L, so a smallest mass at least η and a smallest distance at least L meet the hypothesis of Lemma S23, which gives a unique optimum $\{z_1, \dots, z_k\}$.

Proof of Proposition 6, selected offers. Optimal selection. Let $\{a, b\}$ be an offered pair and c such that $T = \{a, b, c\}$ is equilateral. Every selected offer lies in $\mathrm{Eq}(\mu)$ by Lemma S24. Its cells have positive mass, the members of the cell of a prefer a to b, and those of the cell of b prefer b to a. So $D_{ab} \subset (0, 1)$, and likewise $D_{bc}, D_{ca} \subset (0, 1)$, computed on $\mathcal{D}_\sigma$. Each distribution of Lemma S25 has a single optimal set, so it is selected with the intended offer, whatever the tie-breaking. Hence $V_T \supseteq \mathcal{P}^\circ$ on $\mathcal{D}_\sigma$, and T is offered. For $w \in (0, 1)$, the point $(w, \frac{1}{2}, \frac{1}{2})$ has coordinates in $(0, 1)$ and sum in $(1, 2)$, so it lies in $\mathcal{P}^\circ$; likewise for $(\frac{1}{2}, w, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, w)$. The sets $\{w, \frac{1}{2}\} \times \{\frac{1}{2}\} \times \{\frac{1}{2}\}$ and their rotations are products contained in V_T, they all contain h, and their factors cover $(0, 1)$, hence D_{ab}, D_{bc} and D_{ca}. So T is a star triple.

Selections that recognize obvious clusters. The argument is the same. The distributions of Lemma S25 are clusters in the sense of Definition S4, with L and η independent of ρ, and $Z \in \mathrm{Eq}(\mu)$, since by Lemma S23 the cells of Z are the clusters and their barycentres are the platforms. For ρ also below $\bar{\rho}(\eta, L)$, they lie in $\mathrm{dom}(\sigma)$ and are selected with the offer Z. And $D_{ab} \subset (0, 1)$ for every selection, since equilibrium offers have cells of positive mass. In both cases Theorem 1 applies. □

It suffices that σ recognize clusters on the distributions of Lemma S25: clusters around an equilateral triangle and around far parties.

The minimum-size model admits a similar relaxation of its set of offers.

Remark S10 (Free offers). The proof of Proposition 7 (Appendix B) uses only three offers per triple, one for each choice of the small party, and their far points serve only to give each far party members who arbitrate. A weaker hypothesis on the set $\mathcal{O}$ of admissible offers is therefore enough. For an offer $O \supseteq T$, let V_p be the cell of $p \in O$ relative to O. Call O *free for T, with small party $s \in T$*, if (L1) for every $t \in T$, V_t meets r_t; (L2) each $p \in O \setminus T$ can be assigned to some $t(p) \in T \setminus \{s\}$ such that V_p meets $r_{t(p)}$; and (L3) writing j_t for the number of points assigned to t, $(1 + j_t)\varepsilon \leq 1/2$ for the two $t \neq s$. If every offered pair belongs to a non-aligned triple T such that, for each $s \in T$, $\mathcal{O}$ contains an offer free for T with small party s, then no rule satisfies anonymity, independence, non-imposition and non-nullness. Indeed, take $s = a$. The interval J of Step 2 of that proof is non-empty by (L3) and $k\varepsilon \leq 1$. For each $p \in O$, pick $y_p \in V_p \cap r_t$, with $t = p$ if $p \in T$ and $t = t(p)$ otherwise. Since V_p is open and r_t lies inside the cell of t relative to T, a small disc centred at y_p lies in V_p and in that cell, and the line of r_t cuts it in two. Putting the mass of the cell of p on this disc, split freely between the two halves, lets the members of p rank t first in T and arbitrate freely between the two others. No member of another party lies in the cell of a relative to T, so $m_a = \varepsilon$. Steps 4 and 5 of that proof then go through unchanged, and the cells have masses $n_t \geq \varepsilon$ and ε. Far parties are not needed: if T is equilateral, $t \in r_t$, so (L1) holds for every $O \supseteq T$, and the other parties may sit on the segments from ω to the vertices, inside the triangle.

S4 Weak non-imposition and edge rules

This section proves the fourth variant of Section 8 of the paper. The notation is that of Appendix C: a triple, share vectors in cyclic order, $V \subset \mathbb{R}^3$ convex and non-empty with $D_i = \pi_i(V)$, a rule as a triple of labellings $\ell_i : D_i \to \{\mathsf{F}, \mathsf{S}, \mathsf{I}\}$, and the thirteen admissible patterns.

Definition S5 (Non-imposition and non-nullness on a triple)**.** A rule on the triple may satisfy the following properties.

(i) *Weak robust non-imposition*: for each i, the set of $u \in D_i$ with $\ell_i(u) \in \{\mathsf{F}, \mathsf{I}\}$ contains a non-degenerate interval, and so does the set of $u \in D_i$ with $\ell_i(u) \in \{\mathsf{S}, \mathsf{I}\}$.

(ii) *Strong robust non-imposition* (Axiom 5): for each i, the set $\{\ell_i = \mathsf{F}\}$ contains a non-degenerate interval, and so does $\{\ell_i = \mathsf{S}\}$.

(iii) *Non-nullness*: some $\ell_i(u)$, $u \in D_i$, is F or S.

(iv) *Robust non-nullness*: for some i, the set $\{\ell_i \in \{\mathsf{F}, \mathsf{S}\}\}$ contains a non-degenerate interval.

Strong robust non-imposition implies (i), (iii) and (iv); (iv) implies (iii).

An *edge rule* is a rule that is valid at every point of V, satisfies weak robust non-imposition and non-nullness, and has $\ell_i = \mathsf{I}$ on $\operatorname{int} D_i$ for every i.

Theorem S10. *Let $V \subset \mathbb{R}^3$ be convex and non-empty, with $D_i = \pi_i(V)$.*

(a) If V has empty interior, the following are equivalent: some rule is valid at every point of V and satisfies weak robust non-imposition and non-nullness; a valid quota rule exists; no D_i is a single point.

(b) If V has non-empty interior, let ℓ be a rule that is valid at every point of $\operatorname{int} V$, satisfies weak robust non-imposition, and labels F or S at least one value interior to some D_i. Then there is a quota rule valid on all of V.

Proof. (a) If some D_i is a point, no rule satisfies weak robust non-imposition, since D_i contains no non-degenerate interval, and no quota rule exists. Otherwise Theorem 5(a) gives a valid quota rule, which has the required properties by Proposition 8.

(b) The proof of Theorem 5(b) uses strong robust non-imposition at one point only, in Lemma 13, to show that each set K_i of change points is non-empty; everything else (the symmetries, Lemmas 11, 12, 14, 15, 16 and 17) uses only validity on $U = \operatorname{int} V$. Under the present hypotheses, the two lemmas below give $K_i \neq \emptyset$, and the rest of the proof applies verbatim. The notation (U, E_i, fibres, K_i) is that of the paper. □

Lemma S26 (All or nothing for indifference)**.** *Under the hypotheses of Theorem S10(b), no point of U has the pattern $(\mathsf{I}, \mathsf{I}, \mathsf{I})$.*

Proof. Let Z be the set of points of U with pattern $(\mathsf{I}, \mathsf{I}, \mathsf{I})$. Take $v \in U$ and a small cube $Q \subset U$ centred at v; it is a product of intervals. Suppose Q contains a point of Z. Then $\mathsf{I} \in \Lambda_i$ for every i. If some Λ_i had two elements, Lemma 11 would make the two others strict, contradicting $\mathsf{I} \in \Lambda_j$. So $\Lambda_i = \{\mathsf{I}\}$ for every i, and $Q \subset Z$; in particular $v \in Z$. Taking $v \in Z$, this shows that Z is open. Taking v a limit of points of Z, it shows that Z is closed in U. Since U is connected, $Z = \emptyset$ or $Z = U$. If $Z = U$, then $\ell_i = \mathsf{I}$ on $E_i = \pi_i(U)$ for every i, which the hypothesis excludes. □

Lemma S27 (The change points exist, weak version)**.** *Under the hypotheses of Theorem S10(b), K_1, K_2 and K_3 are non-empty.*

Proof. Suppose $K_1 = \emptyset$. By Lemma 12, ℓ_1 equals a constant λ on $E_1 = \operatorname{int} D_1$. If $\lambda = \mathsf{F}$, the set $\{\ell_1 \in \{\mathsf{S}, \mathsf{I}\}\}$ lies within the two endpoints of D_1 and contains no non-degenerate interval, so weak robust non-imposition fails. The same holds if $\lambda = \mathsf{S}$. So $\lambda = \mathsf{I}$. By Lemma S26, no point of U has the pattern $(\mathsf{I}, \mathsf{I}, \mathsf{I})$, so every point of U has (ℓ_2, ℓ_3) equal to (F, S) or (S, F). If K_2 contained a point w, take $v \in U$ with $v_2 = w$; Lemma 14, applied to coordinate 2, would make $\ell_1(v_1)$ strict, against $\ell_1 = \mathsf{I}$ on E_1. So $K_2 = \emptyset$, and ℓ_2 is constant on E_2, with a strict value since (ℓ_2, ℓ_3) is (F, S) or (S, F) at every point of U. Weak robust non-imposition then fails for coordinate 2, as above. Hence $K_1 \neq \emptyset$, and likewise $K_2, K_3 \neq \emptyset$. □

Corollary S5. *Let $V \subset \mathbb{R}^3$ be convex and non-empty, with $D_i = \pi_i(V)$. The following are equivalent: a valid quota rule exists; a rule exists that is valid at every point of V and satisfies strong robust non-imposition; a rule exists that is valid at every point of V and satisfies weak robust non-imposition and robust non-nullness.*

Proof. A quota rule has the two properties (Proposition 8), and the strong axiom implies the weak one and robust non-nullness. Assume the last condition. If V has empty interior, weak robust non-imposition forces each D_i to contain a non-degenerate interval, and Theorem S10(a) gives a quota rule. Otherwise the interval of robust non-nullness contains values interior to D_i labelled F or S, and Theorem S10(b) gives one. □

Under weak robust non-imposition the equivalence fails. A rule may be indifferent at every interior share and strict only at extreme ones, and such a rule can exist without a quota rule.

Proposition S14 (An edge rule without quota)**.** *Let $T \subset \mathbb{R}^2$ be the triangle with vertices $(0.4, 0.4)$, $(0.6, 0.6)$, $(0.5, 0.4)$, and $V = T \times [0.4, 0.6]$. Then V is a convex polytope in $[0,1]^3$ with $1 \le v_1+v_2+v_3 \le 2$, a rule exists that is valid at every point of V and satisfies weak robust non-imposition and non-nullness, and no valid quota rule exists, whatever the orientations and conventions.*

Proof. The set. $T = \{(x,y) : y \ge 0.4,\ x \ge y,\ x \le 0.5 + (y-0.4)/2\}$. The sums $v_1+v_2+v_3$ lie in $[1.2, 1.8]$. The three projections are $[0.4, 0.6]$. The key point: in T, the value $x = 0.6$ is reached only at $(0.6, 0.6)$, and so is the value $y = 0.6$. So in V, $v_1 = 0.6$ if and only if $v_2 = 0.6$.

The rule. Let $\ell_1(0.6) = \mathsf{F}$, $\ell_2(0.6) = \mathsf{S}$, and I everywhere else; in particular $\ell_3 = \mathsf{I}$. At the points where $v_1 = 0.6$, also $v_2 = 0.6$, and the pattern is $(\mathsf{F}, \mathsf{S}, \mathsf{I})$, which is admissible (it is $a \sim c \succ b$). At the other points the pattern is $(\mathsf{I}, \mathsf{I}, \mathsf{I})$. The rule is non-null. For weak robust non-imposition, $\{\ell_1 \in \{\mathsf{F}, \mathsf{I}\}\} = D_1$ and $\{\ell_1 \in \{\mathsf{S}, \mathsf{I}\}\} = [0.4, 0.6)$ contain non-degenerate intervals; the same holds for ℓ_2; and $\ell_3 = \mathsf{I}$.

No quota rule. Suppose a valid quota rule exists. Since $t_3 \in (0.4, 0.6)$, the values $z = 0.4$ and $z = 0.6$ lie on either side of t_3: one is labelled F, the other S. Above each $(x,y) \in T$, V contains both. To avoid $(\mathsf{F}, \mathsf{F}, \mathsf{F})$ and $(\mathsf{S}, \mathsf{S}, \mathsf{S})$, the rule must give x and y opposite labels: $\ell_2(y)$ is the opposite of $\ell_1(x)$ for every $(x,y) \in T$. Take y with $\max(0.4,\ 2t_1 - 0.6) < y < t_1$, which is possible since $0.4 < t_1 < 0.6$. The slice of T at height y is $[y,\ 0.5 + (y-0.4)/2]$, and it contains a value $x < t_1$ and a value $x' > t_1$. These have different labels ℓ_1, yet both must be the opposite of $\ell_2(y)$: a contradiction. □

The mechanism is as follows. Weak robust non-imposition asks that each direction of each pair win *weakly* on an interval, and a rule indifferent almost everywhere satisfies it at no cost. Such a rule needs to be transitive only at the few points where it is strict. Here the two extreme faces $\{v_1 = \max D_1\}$ and $\{v_2 = \max D_2\}$ coincide, and putting $a \succ b$ and $c \succ b$ on them is transitive. A quota rule, in contrast, is strict everywhere. The full range of s_{ca} above each point of T forces it to make the two other labels opposite on all of T, which a full triangle forbids.

More generally, an edge rule exists when there are $i \ne j$ and endpoints e_i of D_i, e_j of D_j with $V \cap \{v_i = e_i\} = V \cap \{v_j = e_j\} \ne \emptyset$, and no D_k is a single point: label e_i by F, e_j by S, and everything else by I; the verification is the one above.

Corollary S6 (The weak version)**.** *Let $V \subset \mathbb{R}^3$ be convex and non-empty, with $D_i = \pi_i(V)$. A rule that is valid at every point of V and satisfies weak robust non-imposition and non-nullness exists if and only if a valid quota rule exists or an edge rule exists. By Proposition S14, the second alternative cannot be dropped.*

Proof. If V has empty interior, Theorem S10(a) gives a quota rule. Otherwise, let ℓ be such a rule. Either some interior value of some D_i is labelled F or S, and Theorem S10(b) gives a quota rule; or all interior values are labelled I, and ℓ is an edge rule. The converse follows from Proposition 8 and the definition of an edge rule. □

Under Wilson's non-imposition alone, even this weaker equivalence fails.

Example S6 (Wilson's axiom alone)**.** Let T be the triangle of Proposition S14 and $V = \operatorname{conv}\left(T \times [0.4, 0.6] \cup \{(0.5, 0.6, 0.6)\}\right)$, so that $D_i = [0.4, 0.6]$ for each i. Let $\ell_1 = \mathsf{F}$ at 0.6 and S elsewhere, $\ell_2 = \mathsf{S}$ at 0.6 and F elsewhere, $\ell_3 = \mathsf{S}$ at 0.4 and F elsewhere. This rule is strict everywhere, valid on V, and non-imposed in Wilson's sense, but no quota rule and no edge rule is valid on V.

Validity. In V, $v_1 = 0.6$ only at points of $T \times [0.4, 0.6]$ above $(0.6, 0.6)$, so $v_1 = 0.6$ implies $v_2 = 0.6$, and the pattern there is $(\mathsf{F}, \mathsf{S}, \cdot)$. If $v_2 = 0.6$ and $v_1 < 0.6$, the point puts positive weight on the apex $(0.5, 0.6, 0.6)$ and the rest on points above $(0.6, 0.6)$; so $v_3 > 0.4$, and the pattern is $(\mathsf{S}, \mathsf{S}, \mathsf{F})$. Elsewhere it is $(\mathsf{S}, \mathsf{F}, \cdot)$. All are admissible.

No quota rule. V contains $T \times [0.4, 0.6]$ and has the same projections, and the argument of Proposition S14 uses nothing else.

No edge rule. Each of the faces $\{v_2 = 0.4\}$, $\{v_2 = 0.6\}$, $\{v_3 = 0.4\}$, $\{v_3 = 0.6\}$ of V contains a point whose two other coordinates are interior, for instance $(0.45, 0.4, 0.5)$, $(0.55, 0.6, 0.5)$, $(0.45, 0.42, 0.4)$ and $(0.45, 0.42, 0.6)$; there the pattern of an edge rule is (label, I, I), so the label must be I. Hence $\ell_2 = \ell_3 = \mathsf{I}$ everywhere. The faces $\{v_1 = 0.4\}$ and $\{v_1 = 0.6\}$ then carry the pattern (label, I, I) as well, so $\ell_1 = \mathsf{I}$ everywhere, and the rule is null.

Remark S11 (The hypothesis $D_i = \pi_i(V)$)*.* Theorem 2 reads non-imposition and the thresholds on the projections of V. The hypothesis matters when a pair of the triple is also offered elsewhere, so that its set of producible shares D_{ab} is larger than the projection of V. It is used in Lemma 13, where $D_i \setminus E_i$ consists of the endpoints of D_i, and in placing the thresholds in $E_i = \operatorname{int} D_i$. Without it, the route of the proof breaks. Let $V = \{v \in [0.4, 0.6]^3 : v_1 + v_2 \leq 1\}$, convex with non-empty interior, and suppose the pair $\{c, a\}$ is also offered alone, with shares filling $[0, 0.4)$, so that its producible shares form $D_3 = [0, 0.6]$. Let $\ell_1 = \mathsf{F}$ if $v_1 > 0.5$ and S otherwise; ℓ_2 alike; $\ell_3 = \mathsf{F}$ on $[0.4, 0.6]$ and S on $[0, 0.4)$.

1. The pattern $(\mathsf{F}, \mathsf{F}, \cdot)$ would need $v_1 + v_2 > 1$, which is excluded in V, and coordinate 3 is F at every point of V. So every pattern on V is admissible.

2. Strong robust non-imposition holds: inside V for coordinates 1 and 2, and thanks to the two-party offer for coordinate 3.

3. But ℓ_3 is constant on $(0.4, 0.6) = \pi_3(\operatorname{int} V)$, so $K_3 = \emptyset$, against Lemma 13. No quota rule has thresholds in $K_1 \times K_2 \times K_3$.

In this example the rule itself is a quota rule, with $t_1 = t_2 = 0.5$ and $t_3 = 0.4 \in \operatorname{int} D_3$ with the convention $\geq$; only the route of the proof breaks. I do not know whether the equivalence of Theorem 2 survives without the hypothesis.

S5 Four parties: a set of full dimension

Appendix C.3 of the paper gives a convex set V of share vectors on four parties that carries a rule with a band and no quota rule, and its contraction V_1, which citizens produce. Two complements.

The set V is not produced by citizens. In barycentric coordinates, $s_{01} + s_{12} + s_{20} = \frac{1}{3}\big((3 - d) + (2 - 2a - d) + (3 - b - 2c)\big) = 2 + \frac{b}{3}$, which exceeds 2 on the relative interior

of V, where $b > 0$. Every citizen with a strict ranking agrees with at most two of the three duels of the cycle $0 \to 1 \to 2 \to 0$, so the share vectors of any distribution satisfy $s_{01} + s_{12} + s_{20} \le 2$. No point of $\operatorname{relint} V$ is therefore produced by any distribution of citizens, which is why the paper contracts it. On the part of V that citizens can produce, which lies in the face $b = 0$, a quota rule is valid: in the units S_p and the order $01, 02, 03, 12, 13, 23$, thresholds $(\frac{149}{50}, \frac{3}{50}, \frac{79}{40}, \frac{103}{150}, \frac{43}{60}, \frac{38}{25})$, orientations $(+,+,+,+,+,-)$ and the convention $\ge$ on every pair (checked in exact arithmetic). This does not save the characterization, since V_1 is produced and carries the same rule.

Proposition S15 (A set of full dimension)**.** *Let V_2 be the convex hull of the seven points*

$$\tfrac{1}{8}(5,4,4,5,4,3), \quad \tfrac{1}{8}(4,5,5,5,5,3), \quad \tfrac{1}{8}(3,3,5,5,3,4), \quad \tfrac{1}{8}(4,4,4,4,4,4),$$
$$\tfrac{1}{8}(5,3,4,3,5,3), \quad \tfrac{1}{8}(4,4,4,5,4,4), \quad \tfrac{1}{8}(4,3,3,4,5,4),$$

which has non-empty interior in $\mathbb{R}^6$. The rule

$$0 \succ 1 \iff s_{01} \notin (\tfrac{11}{24}, \tfrac{7}{12}], \qquad 0 \succ 2 \iff s_{02} \le \tfrac{7}{12}, \qquad 0 \succ 3 \iff s_{03} \ge \tfrac{11}{24},$$
$$1 \succ 2 \iff s_{12} \ge \tfrac{5}{12}, \qquad 1 \succ 3 \iff s_{13} > \tfrac{5}{12}, \qquad 2 \succ 3 \iff s_{23} \le \tfrac{5}{12},$$

each pair being ranked strictly the other way otherwise, is valid at every point of V_2 and satisfies strong robust non-imposition; it has a band on pair 01*. No quota rule, and no monotone rule with strong robust non-imposition, is valid at every point of* $\operatorname{relint} V_2$*. Every point of V_2 is produced in the plane by the four platforms of Proposition 10. The validity of the rule and the absence of a quota rule are computer-assisted.*

Proof. Validity and the absence of a quota rule are certified as in Proposition 9, by exact rational linear programming and 153 Motzkin certificates over the 32 classes of orientations. Monotone rules and producibility are handled as in the proof of Proposition 10: the vertices of V_2 have coordinates in $[\frac{3}{8}, \frac{5}{8}]$, and each is an exact rational convex combination of the share vectors of the eighteen rankings that the four platforms realize. □

A grid is no substitute for the exact computation. On grids of step $\frac{1}{12}$ to $\frac{1}{36}$, a satisfiability solver finds "quota rules" for the set V of Proposition 9, but they have cyclic points in the continuum, in regions too thin to contain a grid point. The certificates and scripts are available from the author.

S6 Four parties: two families without a rule

This section completes Remark 3(iii) of the paper. Parties are numbered $1, \dots, 4$, and V is the set of share vectors of the societies whose rankings lie in the family. For a label λ, $-\lambda$ is the opposite label ($-\mathsf{I} = \mathsf{I}$), so that $\ell_{yx}(1-s) = -\ell_{xy}(s)$. A pattern on a cycle is admissible if and only if it is $(\mathsf{I}, \mathsf{I}, \mathsf{I})$ or contains an F and an S, so replacing every label by its opposite preserves validity and strong robust non-imposition. Under this axiom each ℓ_p takes both strict values on $(0,1)$, and so has a change point there.

Proposition S16 (Two families)**.** *Let*

$$F_1 = \{1234, 1324, 2413, 3412\}, \qquad F_2 = \{1234, 1342, 2341, 4213\}.$$

Neither family carries a rule valid on $\operatorname{relint} V$ *with strong robust non-imposition, whatever its breakpoints.*

Proof. The family F_1. Let $a, b, c, d > 0$ be the masses on the four rankings, in this order, and put $x = c$, $y = d$, $w = a + c$. Then $s_{12} = 1 - x$, $s_{42} = y$, $s_{41} = x + y$, $s_{23} = w$, $s_{31} = y$, $s_{34} = 1 - x$, and the points of relint V are given by $x, y > 0$ and $x < w < 1 - y$. Write $A(x) = \ell_{12}(1 - x)$, $B(y) = \ell_{24}(1 - y) = -\ell_{42}(y)$, $C(z) = \ell_{41}(z)$. On the triple $\{1, 2, 4\}$, with coordinates $(s_{12}, s_{24}, s_{41}) = (1-x, 1-y, x+y)$, the pattern $(A(x), B(y), C(x+y))$ is admissible for all $x, y > 0$ with $x + y < 1$.

The triples $\{1, 2, 3\}$ and $\{2, 3, 4\}$ have dimension 3, so Proposition 11(ii) applies to them. Fix a change point τ of ℓ_{23}. Let t be a change point of ℓ_{12}. If $t+\tau > 1$, take $x \in (1-t, \tau)$, $w \in (x, \tau)$ and y small: the coordinates $(s_{12}, s_{23}, s_{31}) = (1 - x, w, y)$ are all below the thresholds, which is impossible. So $t \leq 1-\tau$, and A has no change point on $(0, \tau)$, hence is constant there (Lemma 12). Let t' be a change point of ℓ_{42}. If $t' < 1 - \tau$, take $y \in (t', 1 - \tau)$, $w \in (\tau, 1 - y)$ and x small: the coordinates $(s_{23}, s_{34}, s_{42}) = (w, 1 - x, y)$ are all above the thresholds, which is impossible. So $t' \geq 1 - \tau$, and B is constant on $(0, 1 - \tau)$. Call λ_A and λ_B these values.

Every $z \in (0, 1)$ is $x + y$ with $x < \tau$ and $y < 1 - \tau$. So $(\lambda_A, \lambda_B, C(z))$ is admissible for all z, and since C takes both strict values, λ_A and λ_B are strict and opposite. Replacing all labels by their opposites if necessary, $\lambda_A = \mathsf{F}$ and $\lambda_B = \mathsf{S}$. Take x_1 with $A(x_1) = \mathsf{S}$, so $x_1 \geq \tau$. For $y < 1 - x_1$, which is at most $1 - \tau$, the pattern $(\mathsf{S}, \mathsf{S}, C(x_1 + y))$ forces $C = \mathsf{F}$ on $(x_1, 1)$. Take y_1 with $B(y_1) = \mathsf{F}$, so $y_1 \geq 1 - \tau$. For $x < 1 - y_1$, which is at most τ, the pattern $(\mathsf{F}, \mathsf{F}, C(x + y_1))$ forces $C = \mathsf{S}$ on $(y_1, 1)$. These two statements contradict each other near 1.

The family F_2. Let $a, b, c, d > 0$ be the masses on the four rankings, in this order. Then $s_{12} = s_{14} = a + b =: u$, $s_{24} = a + c =: v$, $s_{13} = 1 - c$, $s_{23} = 1 - b$, $s_{34} = 1 - d$. The pair (u, v) takes all values in $(0, 1)^2$. On the triple $\{1, 2, 4\}$, the pattern in the coordinates $(s_{12}, s_{24}, s_{41}) = (u, v, 1 - u)$ is $(\ell_{12}(u), \ell_{24}(v), -\ell_{14}(u))$. Since ℓ_{24} takes both strict values, $\ell_{12}(u)$ and $-\ell_{14}(u)$ are strict and opposite. So $\ell_{12} = \ell_{14} =: G$, and G is strict on $(0, 1)$. Write $P(c) = \ell_{13}(1-c)$, $P'(b) = \ell_{23}(1-b)$, $Q(d) = \ell_{34}(1-d)$. The following patterns are admissible:

$$\{1, 2, 3\}: \ (G(u), P'(b), -P(c)) \quad \text{for } 0 < b < u, \ 0 < c < 1 - u;$$
$$\{1, 3, 4\}: \ (P(c), Q(d), -G(1 - c - d)) \quad \text{for } c, d > 0, \ c + d < 1.$$

The triple $\{1, 2, 3\}$ has dimension 3, with coordinates $(u, 1 - b, c)$. Let τ be any change point of G. As for F_1, Proposition 11(ii) shows that P' is constant on $(0, \tau)$ and P on $(0, 1 - \tau)$. Indeed, if t is a change point of ℓ_{23} with $1 - t < \tau$, take $1 - t < b < u < \tau$ and c small: the point is below all thresholds. If t' is a change point of ℓ_{31} with $t' < 1 - \tau$, take $c \in (t', 1 - \tau)$, $u \in (\tau, 1 - c)$ and b small: it is above all of them. For b, c small and every u, the pattern $(G(u), \lambda_{P'}, -\lambda_P)$ is admissible, where $\lambda_{P'}$ and λ_P are the values near 0. Since G takes both strict values, $\lambda_{P'} = \lambda_P$, and this value is strict. Replacing all labels by their opposites if necessary, it is F.

Take c_1 with $P(c_1) = \mathsf{S}$. For $u < 1 - c_1$ and b small, the pattern $(G(u), \mathsf{F}, \mathsf{F})$ forces $G(u) = \mathsf{S}$. So $G = \mathsf{S}$ near 0. Let $\tau_0 = \sup\{\xi : G = \mathsf{S} \text{ on } (0, \xi)\}$. Then $0 < \tau_0 < 1$, $G = \mathsf{S}$ on $(0, \tau_0)$, and τ_0 is a change point of G, since G constant near τ_0 would be S there, against the definition of τ_0. So $P = \mathsf{F}$ on $(0, 1 - \tau_0)$. Take d_1 with $Q(d_1) = \mathsf{F}$. For $0 < c < \min(1 - \tau_0, 1 - d_1)$, the pattern $(\mathsf{F}, \mathsf{F}, -G(1 - d_1 - c))$ forces $G(1 - d_1 - c) = \mathsf{F}$. So $G = \mathsf{F}$ on the interval $(\max(\tau_0 - d_1, 0), 1 - d_1)$. This interval is non-empty and its left end is below τ_0, so it contains points of $(0, \tau_0)$, where $G = \mathsf{S}$. This is a contradiction. □

Corollary S7. *Let F be a family of strict rankings of four parties. A rule valid on V, or on* relint V*, with strong robust non-imposition exists if and only if a quota rule valid on V exists, and then a natural one exists.*

Proof. A valid quota rule satisfies the axiom, since its thresholds are interior. Conversely, a rule on V restricts to a rule on relint V. By the axiom no pair is unanimous in F, and by Proposition 11(i) each triple has at most four restrictions, and if it has four, then they are

consecutive on the hexagon or form two opposite edges; the other sets of four, and all sets of five or six, have dimension 3. Relabelling the parties preserves all this; so does reversing every ranking together with the social ranking, which turns ℓ_{xy} into ℓ_{yx} and keeps natural quota rules natural. Up to these operations, 455 families satisfy these conditions. Of these, F_1 and F_2 are excluded by Proposition S16. On the 414 Condorcet domains, Proposition 12(i) gives a natural quota rule. The last step is computer-assisted: the other 39 carry one, found by computation with thresholds at multiples of $\frac{1}{7}$ and one half, and checked in exact arithmetic by two methods (folder `exact_quotas` of the project). □

S7 Separated offers: further remarks

This section completes Appendix D of the paper, whose notation it uses. $T = \{a, b, c\}$ is a non-aligned triple; $\mathcal{D}_T$ is the set of pairs (x, T) such that T is separated for μ_x, and $\mathcal{D}_T^+ \subset \mathcal{D}_T$ the subset in which each party of T is ranked first by a positive mass of citizens. The orders on T are written as words, $W_X^{\uparrow}$ is the set of the four orders in which X is not first, $W_X^{\downarrow}$ the set of the four orders in which X is not last, and $\pi(o)$ is the mass of the citizens whose order is o.

S7.1 Majority with any treatment of ties when every party has members

Proposition 13 of the paper excludes every rule on $\mathcal{D}_T$, through the triple tie, which a window produces only when the middle party of two reversed orders has no member. On $\mathcal{D}_T^+$ the triple tie is not produced. Majority rule still fails there.

Proposition S17 (Fixed triangle, every party with members)**.** *On $\mathcal{D}_T^+$, hence on $\mathcal{D}_T$, majority rule is not valid, whatever the labels $\ell_{ab}(\frac{1}{2})$, $\ell_{bc}(\frac{1}{2})$, $\ell_{ca}(\frac{1}{2})$ in $\{\mathsf{F}, \mathsf{S}, \mathsf{I}\}$ given to exact ties.*

Proof. Fix $X \in T$ and a naming Y, Z of the two others. Take $0 < r < \frac{1}{2}$ and

$$\pi(YXZ) = \tfrac{1}{2}, \qquad \pi(XZY) = r, \qquad \pi(ZXY) = \tfrac{1}{2} - r,$$

three orders of $W_X^{\downarrow}$; this is the profile of the proof of Lemma 23 with every order reversed. Each party is ranked first by a positive mass (Y, X, Z respectively), so by the realization step of the proof of Proposition 13 this profile is produced in $\mathcal{D}_T^+$. It gives $s_{XY} = s_{YZ} = \frac{1}{2}$ and $s_{XZ} = \frac{1}{2} + r$, so $X \succ Z$ strictly. Write $u \succsim_L v$ when the tie label of the pair $\{u, v\}$ ranks u weakly above v. Validity requires that one does not have both $Z \succsim_L Y$ and $Y \succsim_L X$, since this would give $Z \succsim Y \succsim X$, hence $Z \succsim X$, against $X \succ Z$. This holds for every naming (X, Y, Z) of T, so validity requires that there is no chain $Z \succsim_L Y \succsim_L X$ of distinct parties. But $\succsim_L$ is complete on each of the three pairs. Choosing for each pair an orientation $u \to v$ with $u \succsim_L v$ gives a tournament on three vertices, and every such tournament has a path $Z \to Y \to X$ through its three vertices: if it is transitive, its order is such a path; if it is cyclic, any two consecutive arcs form one. This path is a chain $Z \succsim_L Y \succsim_L X$, a contradiction. □

Remark S12. The same profile proves the "only if" of Lemma 23 for $W = W_X^{\downarrow}$ directly, without the reversal used in the paper. Let $\tau \notin W_X^{\downarrow}$: τ is cyclic, or a linear order with X last. Name the two other parties Y, Z so that $Y\,\tau\,X$ and $Z\,\tau\,Y$; this covers $\tau = ZYX$ and the cycle $Z \to Y \to X \to Z$, and the exchange of the names covers $\tau = YZX$ and the cycle $Y \to Z \to X \to Y$. On the profile above the ties give $Y \succ X$ and $Z \succ Y$, and majority gives $X \succ Z$: the cycle $X \succ Z \succ Y \succ X$.

S7.2 Size vectors of the arc mechanism

Proposition 15(iii) of the paper says that on the societies that give every party members, the arc mechanism realizes exactly the size vectors of the open simplex. The next remark describes what happens when empty parties are allowed. Notation is that of Definition 6: $B(c,\rho)$ is the smallest disc that contains the support, and $p_0 = c$, $p_\pm = c + \Lambda\rho\,(1-\cos\theta,\ \pm\sin\theta)$ are the platforms.

Remark S13. The set of realized size vectors is not the open simplex on the whole domain, where empty parties are allowed, and it is not invariant under permutations. The vector $(0,1,0)$ is realized: put mass $\frac{1}{2}$ on each of two small discs internally tangent at $c \pm \rho e_1$ and contained in the cell of p_0. The vectors $(1,0,0)$ and $(0,0,1)$ are never realized. The cell of p_+ lies in the open half-plane $\{y_2 > c_2\}$, and its closure does not contain $c \pm \rho e_1$, which lie in the open cell of p_0. If μ gave mass 1 to the cell of p_+, the support would meet the circle of $B(c,\rho)$ only in the open upper half-circle. If it did not meet the circle at all, it would already lie in a smaller disc. Otherwise this intersection is a compact set on which $\langle y-c, e_2\rangle \geq 2\delta$ for some $\delta > 0$. By compactness there is $\varepsilon > 0$ such that every point of the support with $|y-c| > \rho - \varepsilon$ satisfies $\langle y-c, e_2\rangle > \delta$. Then for $0 < t < \min(2\delta, \varepsilon)$, every point of the support satisfies $|y - c - te_2|^2 = |y-c|^2 - 2t\langle y-c, e_2\rangle + t^2 < \rho^2$ (if $|y-c| > \rho-\varepsilon$) or $|y-c-te_2| \leq \rho - \varepsilon + t < \rho$ (otherwise). The support would lie in a disc of radius less than ρ, against the minimality of $B(c,\rho)$. The same holds for p_-. When moreover $\Lambda \tan\frac{\theta}{2} < 1$, the vector $(\frac{1}{2}, 0, \frac{1}{2})$ is realized by two small discs internally tangent at $c \pm \rho e_2$, since $|c + \rho e_2 - p_+|^2 - \rho^2 = 4\Lambda\rho^2 \sin\frac{\theta}{2}\,(\Lambda \sin\frac{\theta}{2} - \cos\frac{\theta}{2}) < 0$ and $c + \rho e_2$ lies above the axis. In all cases, the hypothesis of the impossibility results that this mechanism violates is free arbitration.

The arc offer is not continuous in the society: moving a mass ε of citizens to a small disc far from the support enlarges the smallest disc, so an arbitrarily small change in total variation moves the three platforms by an arbitrarily large distance. The framework requires neither continuity nor measurability of the offer, so nothing is wrong; but the mechanism is an example of possibility, not a model of behaviour.

S7.3 Unbounded supports

Remark S14 (Unbounded supports). Definition 6 restricts the arc mechanism to distributions with compact support. It extends to all distributions with a finite first moment: when the support is unbounded, let g be the barycentre of μ and $\sigma = \int |y-g|\,d\mu(y) > 0$, and offer $\{g - \sigma e_2,\ g,\ g + \sigma e_2\}$, which is aligned. The offer still depends on μ only, and it determines (g,σ). An aligned offer is separated, and (A) bears on non-aligned triples only, so Proposition 15(i) extends: M_{-e_1} is a strict total order on every offer. Parts (iii) and (iv) concern the arc offers and are unchanged. Part (ii) extends as well: on the aligned offers, each pair is won strictly both ways. The mechanism is covariant under translations and homotheties: the image of μ by $y \mapsto g' + \kappa(y-g)$, $\kappa > 0$, has barycentre g', parameter $\kappa\sigma$, an unbounded support, and the same shares between corresponding platforms, since bisectors are mapped to bisectors. So it suffices to treat one offer with $g = 0$. Let $\alpha > 0$, $\varepsilon \in (0, \frac{1}{6})$ and $r > 0$ small, and let μ_0 put mass $0.6(1-\varepsilon)$ on a disc of radius r centred at $(0,\alpha)$, mass $0.4(1-\varepsilon)$ on a disc of radius r centred at $(0,-1.5\alpha)$, and mass ε on a standard normal distribution. It has a finite first moment, an unbounded support and barycentre 0, and $\sigma(\mu_0) = 1.2\alpha + O(\varepsilon) + O(r)$. For ε and r small, the bisector of 0 and σe_2, the line $y_2 = \sigma/2 \approx 0.6\alpha$, leaves the first disc above it and the second below it, so $s_{\sigma e_2, 0} \geq 0.6(1-\varepsilon) > \frac{1}{2}$. Let $\bar{\mu}_0$ be the image of μ_0 by the reflection $(y_1, y_2) \mapsto (y_1, -y_2)$. It has the same barycentre and the same σ, hence the same offer, and above $y_2 = \sigma/2$ it has only the disc at $(0, 1.5\alpha)$ and a part of the normal mass: $s_{\sigma e_2, 0} \leq 0.4(1-\varepsilon) + \varepsilon < \frac{1}{2}$. By the reflection, $\bar{\mu}_0$ gives $s_{-\sigma e_2, 0} > \frac{1}{2}$ and μ_0 gives

$s_{-\sigma e_2,0} < \frac{1}{2}$. The bisector of $\pm\sigma e_2$ is $y_2 = 0$, so μ_0 gives $s_{\sigma e_2,-\sigma e_2} \geq 0.6(1-\varepsilon) > \frac{1}{2}$ and $\bar{\mu}_0$ gives $s_{\sigma e_2,-\sigma e_2} \leq 0.4(1-\varepsilon) + \varepsilon < \frac{1}{2}$. Since M_{-e_1} follows strict majority off ties, each pair is won strictly both ways. On this extended domain the offer jumps from the arc to the aligned offer when a mass ε with unbounded support is added to a society with compact support.