Online Appendix for *Impartial Resolution of Random Conflicts*

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This Online Appendix collects the technical results the article uses, the results on balancing a specified welfare rule, and the examples that show what each requirement excludes. The article's definitions and Assumption 1 apply throughout, and references to the article are hyperlinked.

O1. PROBABILITY AND SCORE-SPACE PRELIMINARIES

This section establishes three facts. The first proposal wins half of all conflicts, whatever the calibration and the rule. Each person's score distribution can be prescribed freely. And any lower score profile can be reached by reducing bundles.

O1.1. The conflict denominator

Lemma O1. *Let* $(\mathbf{m}, \mathrm{R})$ *have* π*-null social ties, and let* q_n *be the probability that* $\mathbf{x} \succ^{\mathbf{m},\mathrm{R}} \mathbf{y}$ *and the pair is a conflict. Then* $q_n > 0$ *and does not depend on the calibration or the rule, and for every* i

$$\pi(F_i(\mathbf{m}, \mathrm{R})) + \pi(O_i(\mathbf{m}, \mathrm{R})) = q_n,$$
$$f_i(\mathbf{m}, \mathrm{R}) = \frac{\pi(F_i(\mathbf{m}, \mathrm{R}))}{q_n} = \mathbb{P}\big(i \text{ prefers the winner} \,\big|\, \text{conflict}\big).$$

In words, the first proposal wins half of all conflicts whatever the calibration and the rule, and a favor rate is the share of conflicts in which the person prefers the winner.

Proof of Lemma O1. Social ties have probability zero, and so do individual ties $x_i \sim_i y_i$, by Assumption 1 and Fubini's theorem; we discard both null sets. In a conflict with $\mathbf{x} \succ \mathbf{y}$, agent i strictly prefers one of the two proposals. If she prefers $\mathbf{x}$, she is favored; otherwise she is overruled. Thus F_i and O_i partition the event that the first proposal wins a conflict. Whether a pair is a conflict depends only on preferences, not on the calibration or the rule. The conflict event has positive probability: by continuity, allocations near $\mathbf{e}_i(1_\Omega)$ and $\mathbf{e}_j(1_\Omega)$, for any two distinct agents, create a strict disagreement. The product of the two neighborhoods is a nonempty open set of pairs, so full support of π gives it positive probability. Exchanging the two draws leaves π unchanged and swaps which proposal wins. Hence the total conflict probability is $2q_n$, and the probability of a conflict in which i prefers the winner is $2\pi(F_i)$. Dividing the latter by the former gives the stated conditional favor rate. □

O1.2. Prescribing marginal score distributions

For a calibration $\mathbf{m}$, let $M_i(\mathbf{x}) = m_i(x_i)$ and let $\Phi_i(t) = \mu\{\mathbf{x} : M_i(\mathbf{x}) \leq t\}$. The near-dictatorship construction prescribes each person's score distribution. The next two lemmas show that this is possible.

Lemma O2. *For every calibration* $\mathbf{m}$*, each* Φ_i *is an increasing homeomorphism of* $[0, 1]$*; equivalently* ν_i *is atomless with full support.*

In words, every person's score distribution is continuous and gives positive probability to every interval of scores.

Proof. Atomless welfare levels make every level set $\{\mathbf{x} : m_i(x_i) = c\}$ null, so Φ_i is continuous. For $0 \le a < b \le 1$, the inverse image of (a, b) under M_i is relatively open and nonempty: m_i is onto, and every single-recipient allocation is feasible by comprehensiveness. Full support of μ (Assumption 1) gives it positive probability, so $\Phi_i(b) > \Phi_i(a)$. Atomlessness also gives $\Phi_i(0) = 0$, while $\Phi_i(1) = 1$. Thus Φ_i is an increasing homeomorphism of $[0, 1]$. □

Lemma O3. *For any increasing homeomorphisms $G_1, \dots, G_n$ of $[0, 1]$ there is a calibration whose score distributions have distribution functions $G_1, \dots, G_n$.*

In words, rescaling a person's scores can give them any continuous, strictly increasing distribution, whatever the scales of the others.

Proof. Let $\mathbf{m}^0$ be any calibration with score distribution functions Φ_i (homeomorphisms by Lemma O2), put $\varphi_i = G_i^{-1} \circ \Phi_i$ and $m_i = \varphi_i \circ m_i^0$. Each m_i is a continuous representation of $\gtrsim_i$ onto $[0, 1]$, and $\mu\{m_i(x_i) \le t\} = \Phi_i(\varphi_i^{-1}(t)) = G_i(t)$. The relabelings φ_i are chosen separately, so the laws are prescribed independently. □

O1.3. Feasible reductions of welfare scores

The proof of the main characterization builds comparison regions in score space. It needs the feasible score set to be downward comprehensive, so that any lower score profile can be reached; it does not need the set to be invariant under permuting calibrated scores. Downward comprehensiveness follows from shrinking each recipient's own bundle.

Lemma O4 (Reachability)**.** *For every $\mathbf{x} \in \mathcal{X}$ and $\mathbf{d} \in [0, 1]^n$ with $\mathbf{d} \le \mathbf{m}(\mathbf{x})$, there is $\mathbf{z} \in \mathcal{X}$ with $\mathbf{z} \le \mathbf{x}$ and $\mathbf{m}(\mathbf{z}) = \mathbf{d}$.*

Proof. For each k, $t \mapsto m_k(tx_k)$ is continuous on $[0, 1]$, equal to 0 at 0 and to $m_k(x_k)$ at 1, so some s_k has $m_k(s_k x_k) = d_k$; then $\mathbf{z} = (s_1 x_1, \dots, s_n x_n) \le \mathbf{x}$ is feasible by comprehensiveness. □

O1.4. Why a density gives atomless welfare

For finitely many goods, if μ has a full-support density on the feasible set, the pair law $\mu \otimes \mu$ satisfies Assumption 1: full support of μ gives full support of π, and it remains to show atomlessness. To verify atomlessness, fix i and a bundle $\bar{x}$, and let $I = \{x \in X : x \sim_i \bar{x}\}$. For $0 \le s < t$ the translates $I + s\mathbf{1}$ and $I + t\mathbf{1}$ are disjoint: a common point y would give $y - s\mathbf{1} \sim_i \bar{x} \sim_i y - t\mathbf{1}$, although the first bundle exceeds the second in every coordinate. These translates have equal Lebesgue measure and lie in a bounded set, so I is Lebesgue-null. Thus $\{\mathbf{x} : x_i \sim_i \bar{x}\}$ is Lebesgue-null in $\mathcal{X}$ and μ-null by absolute continuity, so $\pi\{(\mathbf{x}, \mathbf{y}) : x_i \sim_i \bar{x}\} = \mu\{\mathbf{x} : x_i \sim_i \bar{x}\} = 0$.

O2. THE SYMMETRIC-PROPOSAL OBSTRUCTION

The two-good example in Section 4.5 uses the canonical feasible set. For each good $g \in \{1, 2\}$, let (A_g, B_g) have density 2 on $\{(a, b) \ge 0 : a + b \le 1\}$, independently across goods. This is the uniform law μ on $\mathcal{X}$. It has a full-support density, so the pair law $\mu \otimes \mu$ satisfies Assumption 1, and μ is invariant under interchanging the two agents. The utility functions $v_A = \min\{A_1, A_2\}$ and $v_B = (B_1 + B_2)/2$ are continuous, strictly increase when every own share increases, and have atomless distributions.

An individual share Z has density $2(1 - z)$ on $[0, 1]$ and $\mathbb{P}(Z > z) = (1 - z)^2$. Independence across goods therefore gives

$$\Phi_A(a) = \mathbb{P}(v_A \le a) = 1 - (1 - a)^4.$$

For $b \geq \frac{1}{2}$, write $Z_g = 1 - B_g$. Each Z_g has density $2z_g$, and $v_B > b$ is equivalent to $Z_1 + Z_2 < 2(1-b) \leq 1$. Hence

$$1 - \Phi_B(b) = \int_0^{2(1-b)} \int_0^{2(1-b)-z_1} 4 z_1 z_2 \, dz_2 \, dz_1 = \frac{8}{3}(1-b)^4.$$

The percentile variables are $U_A = \Phi_A(v_A)$ and $U_B = \Phi_B(v_B)$. Since $\Phi_A(\frac{1}{2}) = 15/16$ and $\Phi_B(\frac{1}{2}) = 5/6$, the event $\{U_A > 15/16,\ U_B > 5/6\}$ requires $A_1, A_2 > 1/2$ and $(B_1 + B_2)/2 > 1/2$. Feasibility makes this impossible.

The reversed event has positive probability. At $A_1 = A_2 = 3/8$ and $B_1 = B_2 = 5/8$, the two percentiles are

$$\Phi_A(3/8) = \frac{3471}{4096} > \frac{5}{6}, \qquad \Phi_B(5/8) = \frac{485}{512} > \frac{15}{16}.$$

Reduce both B shares by a sufficiently small positive amount. The resulting allocation is in the interior of the feasible set and still satisfies both strict inequalities. Continuity then gives an open feasible neighborhood in the reversed event, which has positive probability under the uniform law. Thus $\mathbb{P}(U_A > 15/16, U_B > 5/6) \neq \mathbb{P}(U_A > 5/6, U_B > 15/16)$, proving nonexchangeability. In words, agent A cannot be in her top sixteenth while B is in her top sixth, but B can be in her top sixteenth while A is in her top sixth, although the proposals treat the two agents identically. For two agents, Theorem 5 therefore rules out a rule-independent balanced calibration. The conclusion needs no claim about the size of the imbalance under any particular aggregator.

O3. SAMPLING DESIGN FOR THE ONE-GOOD ECONOMY

The favor rates quoted in Section 4.2 are Monte Carlo estimates, not population identities.

The proposal law is the mixture described in Figure 1; 4,000,000 independent pairs are drawn (seed 2026111303), of which 62% are conflicts. With $m_i(x) = x$, agent 1's favor rates are 0.380 (leximin), 0.620 (leximax), 0.500 (utilitarian) and 0.498 (Nash); the expected rates are within 0.001 of one half, as Theorem 6 requires. Along $m_1(x) = x^\lambda$, $m_2(x) = x$, the leximin favor rate of agent 1 crosses one half at $\lambda \approx 1.10$, where the expected rate is 0.654. These calculations illustrate specified rules under given proposal laws; they do not test the universal conditions in Theorems 5 and 6.

O4. EQUIVALENT INCOME

This section proves Proposition 2 of the article. We use the notation of the proof of Theorem 2 in the article: $U = \min_\omega a_\omega$ and $V = \frac{1}{K}\sum_\omega b_\omega$ for a proposal, with indices x and y for the two proposals, and μ_b for the mean of B's share of a good. The formulas follow from $\min(y, 1) = \min x$ for A and from the equality of average shares for B. Let E_K be the event $V \geq 1 - \frac{1}{K}$, the only case in which $y_B \geq 0$. For large K, $1 - \frac{1}{K} > \mu_b$, and Hoeffding's inequality gives $\mathbb{P}(E_K) \leq e^{-2K(1-1/K-\mu_b)^2}$. Outside E_K, $y_B < 0 \leq y_A$. Leximin sides with A only if $\ell_A < \ell_B$, which requires E_K for both proposals. Leximax rules against A only if $w_B > w_A \geq 0$, which requires E_K for one of them. The utilitarian rule sides with A only if $|U_x - U_y| > K|V_x - V_y|$; since $|U_x - U_y| \leq 1$, this requires $\sqrt{K}|V_x - V_y| \leq K^{-1/2}$, an event of probability $O(K^{-1/2})$ by the Berry–Esseen inequality. Conflicts depend on preferences only, so Lemma 4 bounds the denominators below, under π and, with $\frac{1}{4}$, under $\bar{\pi}$. □

O5. FURTHER RESULTS ON SOCIAL ORDERING FUNCTIONS

This section collects the examples that show what each requirement of Section 3 of the article excludes.

Mixture consistency excludes other rules

For three or more agents, mixture consistency excludes leximin. Take $\mathbf{a} = (0.8, 0.9, 0.5)$, $\mathbf{b} = (0.1, 0.7, 0.9)$, $\mathbf{a}' = (0.9, 0.3, 0.2)$ and $\mathbf{b}' = (0.7, 0.2, 0.3)$. Leximin prefers $\mathbf{a}$ to $\mathbf{b}$ and $\mathbf{a}'$ to $\mathbf{b}'$; in both comparisons agents 1 and 2 prefer the first vector and agent 3 the second; yet the half-mixtures $(0.85, 0.6, 0.35)$ and $(0.4, 0.45, 0.6)$ are ranked the other way. With two agents leximin satisfies mixture consistency, because a two-agent conflict is settled by one linear comparison between the decisive scores (Proposition 6); full continuity excludes it. Sums of a nonlinear transformation fail too. With squares, $(0.4, 0.7)$ is indifferent to $(0.1, 0.8)$ and $(0.9, 0.1)$ to $(0.1, 0.9)$, yet the half-mixture $(0.65, 0.4)$ falls below $(0.1, 0.85)$.

Verification of Remark 1

We verify the claims of the remark for the ranking W^ξ. Both terms rise weakly with each person's preference, the first strictly, which gives (P); (A) and (C*) are immediate. A single-recipient allocation gives everyone else the empty bundle, whose index and percentile are zero, so $W^\xi(\mathbf{e}_i(x)) = U_i^\xi(x)$: positions are compared by percentiles, and (ES) follows as in Theorem 3. The second term does not depend on the proposal law and the first is linear in it, so $W^{\xi^\lambda} = \lambda W^\xi + (1-\lambda) W^{\xi'}$, which gives (MC). Frequency independence fails. Let every agent value only good ω_1, so that $e_i(x)$ is her quantity of it, and let $c = 1/(5n)$. The construction of Lemma 5, applied to agents 1 and 2, gives an environment ξ and allocations $\mathbf{x}, \mathbf{y}$ that give these two agents the quantities $(c, 4c)$ and $(2c, 3c)$, give nothing to anyone else, and have percentile vectors $(0.4, 0.6, 0, \ldots, 0)$ and $(0.6, 0.4, 0, \ldots, 0)$. It also gives an environment ξ' and allocations $\mathbf{x}', \mathbf{y}'$ with the same percentile vectors and the quantities $(2c, 4c)$ and $(3c, c)$. The sums of percentiles tie, so the second term decides: $\mathbf{y} \succ_\xi \mathbf{x}$ because $4c^2 < 6c^2$, and $\mathbf{x}' \succ_{\xi'} \mathbf{y}'$ because $8c^2 > 3c^2$. Since positions are compared by percentiles, frequencies equal percentiles, and the two pairs have the same frequencies.

Ordinal information and continuity

Could ordinal information alone suffice? By Proposition 4, a ranking that used only how positions compare with one another would be unchanged by any common increasing transformation of percentiles.

Proposition O1. *No continuous aggregator ranks score vectors the same way after every common increasing homeomorphism of* $[0, 1]$.

Proof. Let $\mathbf{u} = (\frac{1}{2}, \ldots, \frac{1}{2})$ and $\mathbf{v}^{\eta,t} = (\frac{1}{2} - \eta, t, \frac{1}{2}, \ldots, \frac{1}{2})$ with $0 < \eta < \frac{1}{2} < t \leq 1$. For fixed η, all $\mathbf{v}^{\eta,t}$ with $t > \frac{1}{2}$ stand in the same order relation to $\mathbf{u}$, coordinate by coordinate, so invariance makes the ranking of $\mathbf{u}$ against $\mathbf{v}^{\eta,t}$ constant in t. As $t \to \frac{1}{2}$, $\mathbf{v}^{\eta,t}$ tends to a vector that $\mathbf{u}$ strictly Pareto-dominates, so by continuity the constant ranking is $\mathbf{u} \,\mathrm{P}\, \mathbf{v}^{\eta,t}$. Letting $\eta \to 0$ with $t = 1$, continuity gives $\mathbf{u} \,\mathrm{R}\, (\frac{1}{2}, 1, \frac{1}{2}, \ldots, \frac{1}{2})$, a vector that strictly Pareto-dominates $\mathbf{u}$. □

O6. TRANSFORMED UTILITARIAN RULES

The threshold rules in the class $\mathcal{L}$ of the article are probes. Does a familiar family of objectives reveal the same obstruction? For two agents, consider rankings represented by $\varphi(a_1) + \varphi(a_2)$, where the common transformation φ is continuous and strictly increasing. The ordinary sum uses $\varphi(t) = t$; a concave transformation such as $\varphi(t) = \sqrt{t}$ gives more priority to improvements at low scores. Every such rule is anonymous and respects Pareto improvements.

Theorem O1. *Let $n = 2$. A calibration is balanced for every aggregator represented by $\varphi(a_1) + \varphi(a_2)$, with φ continuous and strictly increasing on $[0, 1]$, for which the favor rates are defined, if and only if condition (d) of Theorem 5 holds: it is a common increasing transformation of percentiles, and $(U_1(X_1), U_2(X_2))$ has the same law as $(U_2(X_2), U_1(X_1))$.*

The result relies on variation across the family, not on the ordinary sum alone. A transformation that rises steeply near a chosen threshold makes the sum sensitive to who crosses it. Requiring balance for every such transformation therefore reveals who holds which value within the ordered score pair. The proof approximates the threshold comparisons by such sums without creating social ties (Section O6.1). The theorem characterizes a family of continuous welfare functions; it is not a test with one utilitarian objective.

O6.1. Proof of Theorem O1

Theorem O1 requires balance across the *whole* family of generalized utilitarian rules with a common continuous increasing transformation. The proof shows why testing this family suffices: it approximates threshold-count comparisons within the family and recovers who holds each score. A transformation that rises sharply when a score crosses a threshold makes the rule approximately count the agents above that threshold. We first show that balance passes to these count rules. Comparing four of them then reveals which named agent holds the larger score in every relevant region of ordered profiles. Full support makes these comparisons informative enough to identify the entire off-diagonal score law.

If. Under the conditions of Theorem 5, ν is positionally symmetric, and (8) makes $\pi(F_i)$ and $\pi(O_i)$ independent of i under every aggregator.

Only if. Let $n = 2$, and write $\mathbf{a} = \mathbf{m}(\mathbf{x})$ and $\mathbf{b} = \mathbf{m}(\mathbf{y})$. A score in the first proposal equals a score in the second with probability zero. Cross-proposal ties therefore create no ambiguity when we approximate threshold comparisons.

Step 1: count rules. A threshold count records how many agents reach a stated score: for $\tau \in (0, 1)$ put $N_\tau(\mathbf{a}) = \#\{i : a_i \geq \tau\}$. For example, with scores $(0.2, 0.8)$ exactly one agent reaches a threshold of 0.5. With several thresholds, a rule can give priority to the count at the first threshold, then the second, and so on. Formally, for $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)$ in $(0, 1)$, let $L_{\boldsymbol{\tau}}$ rank $\mathbf{a}$ above $\mathbf{b}$ if $(N_{\tau_1}(\mathbf{a}), \ldots, N_{\tau_k}(\mathbf{a}))$ is lexicographically larger than the corresponding vector for $\mathbf{b}$, or, if the vectors agree, leximin ranks $\mathbf{a}$ above $\mathbf{b}$; $L_\varnothing$ is leximin. Every count rule is anonymous. It satisfies strong Pareto because raising a score weakly increases all threshold counts and leximin decides remaining ties. A social tie under this rule requires a cross-proposal score tie, which has probability zero.

We approximate each count rule by a generalized utilitarian rule. Set

$$\begin{aligned} \varphi_{j,\theta}(z) &= \sum_{l=1}^{k} w_l \, \varsigma_j(z - \tau_l) - e^{-jz} + e^{-j^2} e^{\theta z}, \\ \varsigma_j(s) &= \min\{1, \max\{0, 1 + js\}\}, \\ w_l &= 4^{k-l+1}, \qquad \theta \in [1, 2]. \end{aligned}$$

These functions are continuous and strictly increasing. The step approximations ς_j encode threshold counts, while $-e^{-jz}$ breaks equal-count comparisons according to leximin. The last term is vanishingly small. It lets us choose approximating rules with *null social ties*, which atomless individual scores alone do not guarantee.

To see this, let Q be the law of an ordered score vector. It has no atoms: any fixed ordered vector requires some individual score to take one of finitely many fixed values. For two distinct ordered vectors s, t, the function $\theta \mapsto \sum_i e^{\theta s_i} - \sum_i e^{\theta t_i}$ is nonconstant and real analytic. Indeed, if it were constant on an interval, analyticity would make it constant everywhere and its value at zero would make that constant zero. Its first two derivatives at zero would give equal sums and sums of squares for s and t, forcing the two ordered pairs to coincide. Thus the equation $\sum_i \varphi_{j,\theta}(s_i) = \sum_i \varphi_{j,\theta}(t_i)$ holds for at most a Lebesgue-null set of θ. Fubini's theorem, applied to two independent draws from the non-atomic law Q, yields some $\theta_j \in [1, 2]$ for which $\sum_i \varphi_{j,\theta_j}(a_i)$ is atomless. The corresponding generalized utilitarian rule has null social ties, so the assumed balance applies to it.

For almost every proposal pair no score equals a threshold and no cross-proposal scores coincide. For sufficiently large j the step terms are exact, and a difference in count vectors is decided by its first unequal entry, since $w_l > 2\sum_{l'>l} w_{l'} + 2$. If the counts agree, suppose the two ordered profiles satisfy $s_1 < t_1$. The exponential term ranks t above s, since for all sufficiently large j,

$$\sum_{i=1}^{2} e^{-js_i} - \sum_{i=1}^{2} e^{-jt_i} \geq e^{-js_1} - 2e^{-jt_1} \geq \tfrac{1}{2}e^{-js_1}.$$

The absolute difference of the perturbation terms is at most $4e^{2-j^2} = o(e^{-j})$, uniformly in $\theta \in [1,2]$, whereas $e^{-js_1} \geq e^{-j}$. It cannot reverse this comparison, even though θ_j may vary with j. Distinct minima occur almost surely because a tie would be a cross-proposal score tie. Hence the rankings of the tie-free generalized utilitarian rules converge almost surely to those of $L_{\boldsymbol{\tau}}$. Dominated convergence transfers their equal favorable probabilities to the count rule, whose common conflict denominator is $q_2 > 0$ (Lemma O1).

Step 2: four rules. We now use count rules to detect whether agent 1 or agent 2 holds the larger score in a specified kind of proposal. For two rules R, R' with null social ties, let $\mathcal{D}$ be the set of proposal pairs on which R selects **a** while R' selects **b**. By exchangeability of the *two independent proposals*, the pairs with the opposite change of ranking are reflections of $\mathcal{D}$. Consequently, using G_i from (8),

$$G_i(\mathrm{R}) - G_i(\mathrm{R}') = \mathbb{P}(\mathcal{D},\ a_i > b_i) - \mathbb{P}(\mathcal{D},\ b_i > a_i).$$

Choose $0 < \theta_1 < \theta_2 < \theta_3 < 1$ and compare $\mathrm{R} = L_{(\theta_2,\theta_3)}$ with $\mathrm{R}' = L_{(\theta_2)}$. They disagree only when their counts at θ_2 agree, their counts at θ_3 differ, and leximin ranks the proposals the other way. Up to null sets, $\mathcal{D}$ is the set of pairs with $a_{(1)} < b_{(1)}$ and $b_{(2)} < \theta_3 \leq a_{(2)}$, together with either $\theta_2 \leq a_{(1)}$ or $b_{(1)} < \theta_2 \leq b_{(2)}$. To verify that these cases exhaust the disagreement set, note that leximin must prefer **b**, so $a_{(1)} < b_{(1)}$. The only possible strict count advantage for **a** at θ_3 is then one versus zero: if **a** had two scores above θ_3, so would **b**. Equal counts at the lower threshold θ_2 can be two, requiring $\theta_2 \leq a_{(1)}$, or one, requiring $b_{(1)} < \theta_2 \leq b_{(2)}$. They cannot be zero, since $a_{(2)} \geq \theta_3 > \theta_2$. On this set, agent i prefers **x** precisely when she holds the larger coordinate of **a**; otherwise she prefers **y**. Balance under both rules therefore gives

$$\mathbb{P}(\mathcal{D},\ \text{agent 1 holds } a_{(2)}) = \mathbb{P}(\mathcal{D},\ \text{agent 2 holds } a_{(2)}).$$

Compare next $\mathrm{R} = L_{(\theta_1,\theta_2,\theta_3)}$ with $\mathrm{R}' = L_{(\theta_1,\theta_2)}$. The disagreement set is the previous one except that pairs with $a_{(1)} < \theta_1 \leq b_{(1)} < \theta_2 \leq b_{(2)} < \theta_3 \leq a_{(2)}$ are removed, since the count at θ_1 now determines their ranking. No new disagreement is created: if the counts at θ_1 differ, both new rules make the same choice; if they agree, the comparison is unchanged. In the two-count case above they both equal two at θ_1. In the one-count case they differ precisely when $a_{(1)} < \theta_1 \leq b_{(1)}$. This verifies the stated removal, including proposals whose two scores coincide. To record which agent owns the higher score, define for a Borel set E of ordered pairs (u,v) with $u < v$,

$$\rho(E) = \mathbb{P}(\mathbf{a}_{()} \in E,\ a_1 > a_2) - \mathbb{P}(\mathbf{a}_{()} \in E,\ a_2 > a_1).$$

The difference between the two balance identities, using independence of the proposals, is

$$\mathbb{P}\big(\theta_1 \leq b_{(1)} < \theta_2 \leq b_{(2)} < \theta_3\big) \cdot \rho\big(\{(u,v) : u < \theta_1,\ v \geq \theta_3\}\big) = 0.$$

Thus whenever the specified comparison region has positive probability, the owner asymmetry ρ vanishes in the corresponding region of ordered profiles.

Step 3: conclusion. We must show that the owner-asymmetry measure ρ is zero, not just that one of its values vanishes. Let $\mathcal{I} = \mathbf{m}(\mathcal{X})$ be the feasible set of score pairs and set $c^* = \max\{c : (c,c) \in \mathcal{I}\}$,

as in the proof of Theorem 5. Then $a_{(1)} < c^*$ almost surely and $\mathcal{I} \supseteq [0, c^*]^2$. Choose $0 < \theta_1 < \theta_3 < 1$ with $\theta_1 < c^*$. The open set $\{(u, v) : \theta_1 < u < v < \min(\theta_3, c^*)\}$ meets $\mathbf{m}(\mathcal{X})$, so full support gives it positive probability. It is covered by the countable family of sets $\{\theta_1 \le u < \theta_2 \le v < \theta_3\}$ over rational $\theta_2 \in (\theta_1, \theta_3)$; at least one has positive probability. The preceding identity then implies $\rho(\{u < \theta_1,\ v \ge \theta_3\}) = 0$. The restrictions $\theta_1 > 0$ and $\theta_3 < 1$ lose no probability: $\{u < 0\}$ is empty and $\{v \ge 1\}$ is null by atomlessness. Put $R(r, t) = \rho(\{u < r,\ v \ge t\})$. For $0 < r_1 < r_2 < \min(s_1, c^*)$ and $s_1 < s_2 < 1$, finite additivity gives

$$\rho([r_1, r_2) \times [s_1, s_2)) = R(r_2, s_1) - R(r_1, s_1) - R(r_2, s_2) + R(r_1, s_2) = 0.$$

The same holds with an endpoint at zero or one by the null boundary events just noted. Every point with $0 \le u < v \le 1$ and $u < c^*$ lies in such a rectangle, apart from those null boundaries. Choose a countable rational cover by these rectangles. Within each covering rectangle, all smaller half-open rectangles satisfy the same strict separation, so finite signed-measure uniqueness makes the restriction of ρ zero on its Borel subsets. It follows that $\rho = 0$ on the whole off-diagonal region carrying the score law. Equivalently, conditional on distinct ordered scores, either agent holds the larger score with probability one half. On the diagonal, exchanging agents does not change the score vector. Thus the entire score law ν is exchangeable, equivalent for two agents to positional symmetry, which is condition (c) of Theorem 5; condition (d) follows. □

O7. PUBLIC ALTERNATIVES: WHAT DOES NOT CARRY OVER

Two parts of the paper do not carry over, and we claim neither. The first is the converse half of Theorem 5. Its easy half survives: the argument that positional symmetry implies balance under every anonymous rule uses only the independence of the two draws. So if, given the percentile values that occur together in a response, each group is equally likely to hold each value, every anonymous aggregation of percentile rewards gives the groups equal favor rates. This condition can be tested on data in which several groups rank the same responses. The converse, however, uses the structure of private bundles: lowering one person's score without changing anyone else's requires her own bundle, which a public alternative does not provide. The second is Section 3. With a public alternative, no allocation benefits one person alone, so equal standing has no object, and Theorems 3 and 4 do not apply. What transfers is the guarantee in expectation, together with the fact that a reward model's Borda count is a sum of percentiles.

O8. BALANCING A SPECIFIED WELFARE RULE

When the conditions of Theorem 5 fail, no calibration balances every anonymous rule, but a particular rule may still be balanced. A planner who has settled on leximin can choose the scales for leximin alone. This section shows how, with the same meaning of balance.

O8.1. Two agents

With two agents, a resolved conflict favors exactly one of them, so balance means that each wins half the time. Theorem 1 provides calibrations that favor either agent almost always. To move between these extremes, favor rates must vary continuously with the scales. The regularity we need concerns the stability of comparisons, not a continuous representation of the social ordering.

Regularity (M). *For every calibration* $\mathbf{m}$*,* π*-almost every pair* $(\mathbf{x}, \mathbf{y})$ *is* stable*:* $\mathbf{m}(\mathbf{x})$ *and* $\mathbf{m}(\mathbf{y})$ *are not socially indifferent, and the strict social ranking of* $\mathbf{m}(\mathbf{x})$ *and* $\mathbf{m}(\mathbf{y})$ *is the same for all pairs of index profiles in a neighborhood of* $(\mathbf{m}(\mathbf{x}), \mathbf{m}(\mathbf{y}))$.

In words, social ties have probability zero, and almost every strict comparison survives a small change in scores. Leximin satisfies (M): the lowest scores of two independent proposals almost surely

differ, and a small perturbation does not reverse their order. The next theorem therefore applies to leximin without any condition on how percentiles depend on each other within a proposal.

Theorem O2. *Let $n = 2$ and let* R *satisfy* (M). *For every $\phi \in (0,1)$ there is a calibration $\mathbf{m}$ with $f_1(\mathbf{m}, \mathrm{R}) = \phi$. If* R *is leximin, fixing any base calibration $\mathbf{m}^0 = (m_1^0, m_2^0)$, the calibrations $\mathbf{m}^\lambda = ((m_1^0)^\lambda, m_2^0)$, $\lambda \in (0,\infty)$, have $f_1(\mathbf{m}^\lambda, \mathrm{R})$ continuous and nondecreasing in λ, with $f_1(\mathbf{m}^\lambda, \mathrm{R}) \to 1$ as $\lambda \to \infty$ and $f_1(\mathbf{m}^\lambda, \mathrm{R}) \to 0$ as $\lambda \to 0$; every $\phi \in (0,1)$ is attained along this family.*

For a general rule, the proof joins calibrations favoring opposite agents and applies the intermediate value theorem. For leximin, one exponent suffices: raising agent 1's scores to a larger power lowers them inside $(0,1)$ and gives her more priority. The proof turns this intuition into continuity and monotonicity of her favor rate.

O8.2. A balancing prescription

Exact balance for a specified rule follows at once. It does not require the percentile vector to be exchangeable, so it also covers the economy of Section 4.5, in which no calibration balances every rule.

Corollary O1. *Let $n = 2$ and let* R *satisfy* (M). *A balanced calibration exists. Under leximin, the exponents λ with $f_1(\mathbf{m}^\lambda, \mathrm{R}) = \frac{1}{2}$ in the family of Theorem O2 form a nonempty compact interval.*

Under leximin, the balancing exponent has a direct statistical meaning. Suppose a conflict presents the score profiles $(0.8, 0.4)$ and $(0.3, 0.9)$ under a base calibration. Agent 1 prefers the first proposal and agent 2 the second. Leximin chooses the first, because its lowest score, 0.4, exceeds the lowest score of the second, 0.3. The decisive scores are thus those each agent obtains in her *less-preferred* proposal.

In general, let $\ell_i = \min\{m_i^0(x_i), m_i^0(y_i)\}$ denote these scores; they lie in $(0,1)$ almost surely. Under the power calibration, leximin sides with agent 1 exactly when $\ell_1^\lambda < \ell_2$: the smaller of the two less-preferred scores then belongs to the losing proposal. Proposition 6 (Appendix E of the article) shows that this comparison holds in every strict two-agent conflict. Taking logarithms gives

$$f_1(\mathbf{m}^\lambda, \mathrm{R}) = \mathbb{P}\left(\frac{\log \ell_2}{\log \ell_1} < \lambda \,\middle|\, \text{conflict} \right). \tag{O8.1}$$

Given a conflict, the ratio has no atoms: an atom would be a positive-probability score tie across independent proposals. Every median of the ratio therefore yields $f_1 = f_2 = \frac{1}{2}$. Concretely, the planner computes $\log \ell_2 / \log \ell_1$ in each conflict and adopts the median as the exponent; leximin then sides with each agent half the time.

This is a population prescription, not a guarantee from a fitted sample. An institution can estimate the median on one sample and test the exponent on another. It needs ordinal preferences, base scores and proposal pairs, but no interpersonal measure of gains and losses. The resulting scale balances leximin; it need not balance another rule, nor equalize the agents' marginal score distributions.

O8.3. Three agents and resource competition

With more agents, several favor rates must become equal at once. Theorem O3, stated in Section O10, gives a general existence result for leximin under an explicit boundary condition. For three agents, this condition holds in a familiar family of proposal laws in which agents compete for resources.

Imagine that each good is split at random among the agents, with any remainder left unused. Different goods may be split independently, but the agents' shares of the same good compete for one unit: more for one leaves less for the others.

Formally, for finitely many goods we draw the n assigned shares and the unused remainder of each good from a symmetric Dirichlet distribution (Mosimann, 1962), independently across goods.

For good ω, all $n+1$ parameters equal $\alpha_\omega > 0$. These proposal laws form the *Dirichlet family*; their parameters describe how submitted shares are spread, not physical divisibility. Parameters equal to one give the uniform law. Small parameters favor extreme divisions; large ones concentrate proposals near equal shares. For every member μ, the pair law $\mu \otimes \mu$ satisfies Assumption 1 and treats the agents symmetrically.

Corollary O2. *Let $n = 3$, let* R *be leximin, let Ω be finite, and let μ belong to the Dirichlet family. Then a balanced calibration exists, for every profile of monotone preferences.*

The proof turns resource competition into *negative association* of the percentiles: increasing functions of the shares of disjoint groups of agents are nonpositively correlated. For three agents, this yields the boundary inequalities needed to choose the power exponents jointly.

We have not established the analogous result for four or more agents.

O9. TWO-AGENT BALANCING PROOFS

We establish the probability decomposition and the path continuity used by the attainment theorem and the median prescription; the leximin comparison is Proposition 6 of the article. The theorem's proof then connects calibrations favoring opposite agents. The argument for more agents is separate: it does not apply the two-agent formula repeatedly.

O9.1. Two-agent comparison and continuity results

These results support the attainment theorem and the median prescription. They distinguish unconditional agreement from the conditional favor rate and justify the continuity needed for the intermediate value theorem. The event $\mathcal{A}_i(\mathbf{m}, \mathrm{R})$ is defined in Appendix A of the article. A pair is *unanimous in the first orientation* when both agents prefer the first proposal; let $p = \pi\{x_1 \succ_1 y_1,\ x_2 \succ_2 y_2\}$. In a *conflict in the first orientation*, agent 1 prefers the first proposal and agent 2 the second; let $q = \pi\{x_1 \succ_1 y_1,\ y_2 \succ_2 x_2\}$. The two draws are symmetric, so the reversed orientations have the same respective probabilities.

Proposition O2. *For $n = 2$, with social ties π-null, $f_1(\mathbf{m}, \mathrm{R}) + f_2(\mathbf{m}, \mathrm{R}) = 1$ and*

$$\pi(\mathcal{A}_i(\mathbf{m}, \mathrm{R})) = \tfrac{1}{2} + p + q\, f_i(\mathbf{m}, \mathrm{R}), \qquad i = 1, 2.$$

Most of this probability has nothing to do with calibration. The probability $\pi(\mathcal{A}_i(\mathbf{m}, \mathrm{R}))$ always includes $\frac{1}{2}$ from pairs in which agent i does not prefer the first proposal, and p from unanimous pairs. Only the last term, $q f_i$, records conflicts resolved in her favor. Favor rates remove this common part. Near-dictatorship therefore means $f_1 \to 1$, not merely a large one-sided agreement probability.

Proof of Proposition O2. A pair with $y_i \gtrsim_i x_i$ lies in $\mathcal{A}_i(\mathbf{m}, \mathrm{R})$ vacuously; these have mass $\frac{1}{2}$ since ties are null. Among pairs with $x_i \succ_i y_i$, the unanimous ones (mass p) lie in $\mathcal{A}_i(\mathbf{m}, \mathrm{R})$ by strong Pareto, and the conflicts (mass q) lie in $\mathcal{A}_i(\mathbf{m}, \mathrm{R})$ exactly when i is favored, which is a fraction $f_i(\mathbf{m}, \mathrm{R})$ of them. Adding gives the identity. For the first claim, each strict two-agent conflict favors exactly one agent, so Lemma O1 gives $f_1 + f_2 = 1$. □

The range theorem requires the *probability* of a favorable decision to vary continuously with the welfare scales. Continuity of each agent's score function does not guarantee this: small changes could reverse a positive share of social rankings. Regularity (M) rules that out, because it makes almost every strict comparison stable under small score changes. The lemma combines this stability with dominated convergence to show that the favor rate is continuous.

Lemma O5. *Let $n = 2$, let* R *satisfy* (M)*, and let $(\mathbf{m}^t)_{t \in T}$, T an interval, be calibrations such that $t \mapsto m_i^t(x)$ is continuous for every i and x. Then $t \mapsto f_1(\mathbf{m}^t, \mathrm{R})$ is continuous.*

Proof. Social ties are π-null by (M), so $\pi(F_1(\mathbf{m}^t, \mathrm{R})) + \pi(O_1(\mathbf{m}^t, \mathrm{R})) = q > 0$ for every t, and it suffices to show that $\pi(F_1(\mathbf{m}^t, \mathrm{R}))$ and $\pi(O_1(\mathbf{m}^t, \mathrm{R}))$ are continuous. Let $t_k \to t$. Preferences do not depend on the calibration, and ties $x_i \sim_i y_i$ are π-null. For π-almost every pair, $\mathbf{m}^{t_k}(\mathbf{x}) \to \mathbf{m}^t(\mathbf{x})$ and $\mathbf{m}^{t_k}(\mathbf{y}) \to \mathbf{m}^t(\mathbf{y})$, and the pair is stable at $\mathbf{m}^t$; so for large k the strict social ranking at t_k is the one at t. The indicators of $F_1(\mathbf{m}^{t_k}, \mathrm{R})$ and $O_1(\mathbf{m}^{t_k}, \mathrm{R})$ therefore converge π-almost everywhere, and dominated convergence applies. □

O9.2. Attaining and balancing favor rates

Proof of Theorem O2. To reach any interior rate, we connect calibrations that nearly always favor opposite agents. A convex combination of the two increasing score transformations is still increasing and onto, so every point of the path is an admissible calibration. *General* (M) *rules.* Fix $\varepsilon > 0$. By Theorem 1 there are calibrations $\mathbf{m}^+$ and $\mathbf{m}^-$ with $f_1(\mathbf{m}^+, \mathrm{R}) > 1 - \varepsilon$ and $f_2(\mathbf{m}^-, \mathrm{R}) > 1 - \varepsilon$. Social ties are null by (M), so Proposition O2 gives $f_1(\mathbf{m}^-, \mathrm{R}) < \varepsilon$. Write $m_i^\pm = \psi_i^\pm \circ U_i$, where U_i is the percentile of agent i (Section 4.3) and $\psi_i^\pm$ are increasing homeomorphisms of $[0, 1]$, and put $\mathbf{m}^t = \big(((1-t)\psi_i^- + t\psi_i^+) \circ U_i\big)_i$, $t \in [0, 1]$. A convex combination of increasing homeomorphisms of $[0, 1]$ is one, so each $\mathbf{m}^t$ is a calibration, and $t \mapsto m_i^t(x)$ is continuous. By Lemma O5, $f_1(\mathbf{m}^t, \mathrm{R})$ is continuous in t, below ε at $t = 0$ and above $1 - \varepsilon$ at $t = 1$. The intermediate value theorem attains every $\phi \in [\varepsilon, 1 - \varepsilon]$, and ε is arbitrary.

For leximin, no arbitrary path is needed: varying one power gives an ordered family of relative scales. The two-agent comparison makes the favor rate monotone along that family. *Leximin.* Write $m_i^0 = \psi_i^0 \circ U_i$. Then $m_1^\lambda = (\psi_1^0)^\lambda \circ U_1$ and $m_2^\lambda = m_2^0$, so by Proposition 6, $f_1(\mathbf{m}^\lambda, \mathrm{R})$ is the conditional probability that $g_\lambda(U_1(y_1)) < U_2(x_2)$, with $g_\lambda = (\psi_2^0)^{-1} \circ (\psi_1^0)^\lambda$. Continuity is Lemma O5. For $u < 1$ we have $\psi_1^0(u) < 1$, so $(\psi_1^0(u))^\lambda$ decreases in λ, and $g_\lambda(u)$ is pointwise nonincreasing in λ; hence $f_1(\mathbf{m}^\lambda, \mathrm{R})$ is nondecreasing. As $\lambda \to \infty$, $g_\lambda(u) \to 0$ for every $u < 1$; as $\lambda \to 0$, $g_\lambda(u) \to 1$ for every $u > 0$. Almost surely $U_1(y_1) \in (0, 1)$ and $U_2(x_2) \in (0, 1)$, so the indicator of $\{g_\lambda(U_1(y_1)) < U_2(x_2)\}$ tends to 1, respectively 0, and dominated convergence gives the limits 1 and 0. □

Proof of Corollary O1. For two agents, balance requires $f_1 = f_2 = \frac{1}{2}$ by Proposition O2. Theorem O2, applied with $\phi = \frac{1}{2}$, provides such a calibration. In the leximin power family, $f_1(\mathbf{m}^\lambda, \mathrm{R})$ varies continuously and nondecreasingly from zero to one. Its level set at $\frac{1}{2}$ is therefore nonempty and, by continuity and monotonicity, is a compact interval, possibly a single point. The argument proves existence, not uniqueness of the balancing exponent. □

O10. THE THREE-AGENT EXISTENCE ARGUMENT

The three-agent corollary follows from a conditional existence theorem for any number of agents. We first define the limiting favor rates on the boundary of the power-exponent simplex. We then prove the existence theorem and verify its boundary condition for the three-agent Dirichlet environment. Throughout, conflicts stay defined for the full population; the boundary analysis depends on it.

Favor rates under leximin. Leximin decides by comparing the lowest welfare score in each proposal. Consider a proposal that loses because its lowest score is smaller. The agent holding that score must prefer the winning proposal: otherwise her score in the winner would be no higher than the loser's minimum, contradicting the ranking. The next lemma states this critical-agent observation. By Assumption 1, all score distributions are atomless and the two proposals are independent. Score ties across proposals, and hence leximin social ties, have probability zero. Thus leximin almost surely chooses the proposal with the higher minimum score. Lemma O1 then identifies f_i as the conditional probability that agent i prefers the winner when the pair is a conflict.

Lemma O6. *For π-almost every pair, the agent holding the smallest index of the* losing *allocation (the* critical agent*) strictly prefers the winner. In a conflict the critical agent is therefore favored.*

Proof. Say $\mathbf{x}$ wins, and let a and b hold the smallest indices of $\mathbf{x}$ and $\mathbf{y}$. Then $m_b(y_b) < m_a(x_a) \leq m_b(x_b)$, so b strictly prefers $\mathbf{x}$. The holder of the *winner's* smallest index need not prefer the winner: $\mathbf{x} = (0.5, 0.6)$ beats $\mathbf{y} = (0.9, 0.3)$, but agent 1 prefers $\mathbf{y}$. □

A family of calibrations. We vary the interpersonal scales while preserving every agent's ranking. Start from the percentile scores $U = (U_1, \ldots, U_n)$, and for s in the open simplex $\Delta^\circ = \{s \in (0,1)^n : \sum_k s_k = 1\}$ define $\mathbf{m}^s = (U_1^{s_1}, \ldots, U_n^{s_n})$ and $f_i(s) = f_i(\mathbf{m}^s, \mathrm{R})$, where R is leximin. Increasing s_i lowers agent i's score at every interior percentile, so she more often determines the leximin minimum. Decreasing s_i raises her score toward one. Every positive exponent vector t equals λs for a common factor $\lambda > 0$ and some $s \in \Delta^\circ$. Replacing all scores by their common power λ preserves the leximin ranking. Normalizing the exponents to sum to one therefore loses no power calibration relevant to leximin.

The map $f : \Delta^\circ \to [0,1]^n$ is continuous. Scores depend continuously on s. For almost every pair, the two proposals have different minimum scores, so their leximin ranking does not change under sufficiently small changes in s. Dominated convergence then makes the favor rates continuous.

Behavior at the boundary. Suppose three agents have exponents approaching $(\frac{1}{2}, \frac{1}{2}, 0)$. Agent 3's score approaches one under almost every proposal. Leximin therefore compares the two proposals through agents 1 and 2 only. Yet agent 3 still counts when we decide whether the *full pair* is a conflict. Dropping her from that denominator would change the balance criterion.

Formally, let Δ be the closed simplex with vertices $e_1, \ldots, e_n$. At a point $s \in \Delta$ with support $I = \{i : s_i > 0\} \neq N$, agents outside I are pushed to the top of the scale. Let $\succ_I$ denote leximin applied only to the agents in I, using indices $U_i^{s_i}$. For every agent, including those outside I, define the limiting favor rate

$$f_k^s = \mathbb{P}\big(k \text{ prefers the } \succ_I\text{-winner} \mid \text{the pair is a conflict}\big), \qquad k \in N,$$

where a conflict still means disagreement among the full set of n agents. The next lemma shows that this restricted leximin comparison gives the correct boundary rates.

Lemma O7. *If $s^m \in \Delta^\circ$ and $s^m \to s \in \Delta$ with support $I \neq N$, then $f_k(s^m) \to f_k^s$ for every $k \in N$.*

Proof. For almost every pair, all percentiles lie in $(0,1)$ and the two allocations' smallest indices among agents in I differ. As $s^m \to s$, the indices of agents outside I converge to 1, while those inside converge to values strictly below 1. Consequently, for all sufficiently large m the smallest index of each allocation belongs to I, and the leximin comparison agrees with $\succ_I$. The preferred proposal is thus eventually unchanged almost surely. Non-unanimity and individual preferences do not depend on the calibration. Lemma O1 fixes the positive conditioning probability, so dominated convergence gives the claimed limits for every k. □

At a vertex of the simplex only one exponent remains positive, say $I = \{i\}$. Agent i then determines which proposal wins, so her limiting favor rate is one. Any excluded agent is favored only when her preference agrees with agent i's, conditional on a disagreement in the full population. Along edges with two or more active agents, the same comparison need not favor one active agent all the time. The condition needed for existence is narrower: no excluded agent may have a favor rate at least as high as all the agents still determining leximin.

Condition (S). *For every $s \in \Delta$ with support I, $\varnothing \neq I \subsetneq N$, and every $j \notin I$, $f_j^s < \max_{i \in I} f_i^s$.*

In words, at every boundary point, an agent whose exponent has fallen to zero cannot have the largest limiting favor rate. On a vertex, the condition holds if each outsider sometimes disagrees with the sole active agent, as full support of π (Assumption 1) ensures for any two agents. On an edge with two active agents, the critical agent always prefers the leximin winner, whereas an outsider benefits only when her preferences agree with that winner. Whether this gives an active agent a strictly higher rate depends on the joint distribution of the agents' welfare. Proposition O3 and Lemma O8 establish the needed inequality for the three-agent Dirichlet environment. They do not assert existence for every proposal distribution.

Theorem O3. *Let* R *be leximin, and suppose that condition* (S) *above holds. Then there is* $s^* \in \Delta^\circ$ *with* $f_1(s^*) = \cdots = f_n(s^*)$*: a balanced calibration exists.*

The KKM argument below rests on the boundary condition, not on monotonicity of every agent's favor rate in every exponent. By the KKM lemma, if closed subsets of a simplex cover each face using only sets indexed by that face's vertices, all those sets have a common point (Knaster et al., 1929).

Proof of Theorem O3

Proof. For each agent i, let $A_i = \{s \in \Delta^\circ : f_i(s) \geq f_l(s)\ \forall l\}$: the set of exponent vectors at which agent i is among the most favored. Continuity of the favor-rate functions f_l makes A_i relatively closed in Δ°. Let $\bar{A}_i$ be its closure in Δ. At every interior exponent vector, at least one agent has the highest rate, so the A_i cover Δ° and the closed sets $\bar{A}_i$ cover Δ. This step uses finiteness: $\bigcup_i \bar{A}_i$ is closed and contains the dense set Δ°.

Boundary exclusion. Let s have proper support I and $j \notin I$. By (S), $f_j^s < \max_{i \in I} f_i^s$. Lemma O7 makes this strict inequality persist for all interior exponent vectors sufficiently close to s: otherwise a convergent sequence would contradict its limit. No such vector belongs to A_j, so $s \notin \bar{A}_j$.

KKM and conclusion. For any face $\Delta_I = \operatorname{conv}\{e_i : i \in I\}$ and any $s \in \Delta_I$, the closed sets $\bar{A}_i$ cover Δ, while boundary exclusion rules out membership in $\bar{A}_j$ for j outside the support of s. Thus $\Delta_I \subseteq \bigcup_{i \in I} \bar{A}_i$. The KKM lemma gives $s^* \in \bigcap_i \bar{A}_i$. This point is interior, because every proper face excludes at least one $\bar{A}_j$. There $\bar{A}_i = A_i$ by continuity, so every agent maximizes the favor rate at s^* and all rates coincide. The proof needs only limits from the interior at each boundary point, which Lemma O7 supplies. It does not assume a continuous extension of the rate functions across different boundary faces. □

The KKM proof gives leximin balance once condition (S) holds, but it does not establish (S) for arbitrary preferences and proposal laws. Where can the required boundary inequalities come from? Competition for resources supplies them. When proposals divide scarce resources, giving more to one agent leaves less for others. An excluded agent is then less likely to prefer the proposal selected by the agents whose scores still determine leximin. Negative association formalizes this form of competition, and the next proposition shows that it suffices for three agents.

Competition for resources. If agent 1 receives an unusually generous allocation of a scarce good, agent 2 may be less likely to receive one. The proof needs a stronger version of this negative relationship: increasing events about disjoint groups of agents must not be more likely to occur together than they would under independence. Random variables $Z_1, \ldots, Z_n$ are *negatively associated* if, for every pair of disjoint coordinate groups $A, B \subseteq N$ and every pair of bounded measurable coordinatewise nondecreasing functions g, h, $\operatorname{Cov}(g(Z_A), h(Z_B)) \leq 0$ (Joag-Dev and Proschan, 1983). Unlike ordinary negative covariance between two coordinates, this condition also controls increasing events involving several agents at once. It lets us compare an outsider's preference with the outcome chosen by the insiders' leximin rule.

Proposition O3. *Under leximin, suppose that the percentile vector* $(U_1(x_1), \ldots, U_n(x_n))$ *is negatively associated under* μ*. Then condition* (S) *holds at every* $s \in \Delta$ *whose support has one or two agents. In particular,* (S) *holds when* $n = 3$*.*

To verify (S), we compare how often an outsider and an active agent prefer the allocation chosen by the active agents. First, unanimous pairs give everybody the same contribution, so removing them does not change the inequality. Second, negative association implies that an outsider agrees with the active group's leximin decision with probability at most one half. Third, with one or two insiders, at least one insider agrees with that decision with probability strictly greater than one half. Together these give the required strict inequality on all boundary faces of a three-agent simplex.

The Dirichlet proposal family has exactly the dependence property needed here. For each good, the vector of shares assigned to agents and to unused resources is negatively associated, for any admissible Dirichlet parameters. Different goods are divided independently. Finally, each agent's welfare percentile is a nondecreasing function of the goods she receives. The closure properties of negative association therefore carry the property across goods and through the individual percentile transformations (Lemma O8).

Proof of Proposition O3

Fix a boundary exponent vector s whose support I has one or two agents. For two independent proposals, write $a_k = U_k(x_k)$ and $b_k = U_k(y_k)$. Agent k prefers the first proposal exactly when $a_k > b_k$, up to a null set. Let $W = \{\min_{i\in I} a_i^{s_i} > \min_{i\in I} b_i^{s_i}\}$ denote the event that leximin among the active agents selects $\mathbf{x}$. The restricted rule produces no ties, and exchanging the two independent proposals gives $\mathbb{P}(W) = \frac{1}{2}$. We compare how often insiders and outsiders agree with the winning proposal.

Step 1: remove unanimous comparisons. In a unanimous pair, every agent prefers the same proposal, and strong Pareto makes that proposal the $\succ_I$-winner. Subtracting the common unanimous probability from each agent's unconditional chance of agreeing with the restricted winner leaves

$$f_k^s = \frac{\mathbb{P}(k \text{ prefers the } \succ_I\text{-winner}) - \mathbb{P}(\text{unanimous})}{\mathbb{P}(\text{conflict})}.$$

The denominator is common and positive. Consequently (S) at s is equivalent to

$$\mathbb{P}(j \text{ prefers the winner}) < \max_{i\in I} \mathbb{P}(i \text{ prefers the winner}) \qquad \text{for every } j \notin I.$$

We can therefore work with unconditional probabilities rather than with a conditional conflict event.

Step 2: outsiders. Let $j \notin I$ and $E = \{a_j > b_j\}$ be the event that the outsider prefers the first proposal. The outsider prefers the restricted leximin winner exactly when W and E both occur or both fail. We show that each conjunction has probability at most one quarter.

Conditional on $\mathbf{b}$, the law of $\mathbf{a}$ is unchanged, because the two proposals are independent. We are not conditioning a negatively associated vector on some of its own coordinates. For fixed $\mathbf{b}$, the event W increases with the active agents' scores $(a_i)_{i\in I}$, while E increases with the outsider's score a_j. Negative association of $\mathbf{a}$ therefore implies $\mathbb{P}(W \cap E \mid \mathbf{b}) \le \mathbb{P}(W \mid \mathbf{b})\,\mathbb{P}(E \mid \mathbf{b})$. The first factor decreases with $(b_i)_{i\in I}$ and the second decreases with b_j. Negative association of $\mathbf{b}$, applied to their negatives, therefore gives $\mathbb{E}[\mathbb{P}(W \mid \mathbf{b})\,\mathbb{P}(E \mid \mathbf{b})] \le \mathbb{P}(W)\mathbb{P}(E) = \frac{1}{4}$. Exchanging the proposals yields $\mathbb{P}(W^c \cap E^c) \le \frac{1}{4}$. Thus $\mathbb{P}(j \text{ prefers the winner}) = \mathbb{P}(W \cap E) + \mathbb{P}(W^c \cap E^c) \le \frac{1}{2}$.

Step 3: insiders. With one active agent $I = \{i\}$, the restricted leximin rule always selects her preferred proposal, so her unconditional agreement probability is one. With two active agents $I = \{i, i'\}$, Lemma O6 shows that at least one of them prefers the restricted winner. Both prefer it with positive probability. To see this, note that each percentile is positive almost surely, so there exists a feasible allocation $\mathbf{z}$ at which both active agents have positive percentiles. Both therefore strictly

prefer $\mathbf{z}$ to the zero allocation. Continuity preserves these preferences on a nonempty relatively open set of proposal pairs, and full support gives that set positive probability. This argument requires positive welfare, not a strictly positive quantity of every good. On an active-agent disagreement exactly one insider prefers the winner, whereas on an active-agent unanimous pair both do. Hence their agreement probabilities sum to $1 + \mathbb{P}(\text{the active agents agree}) > 1$. At least one is therefore strictly above one half, whereas every outsider is at most one half.

By Steps 2 and 3, the equivalent form of (S) in Step 1 holds at s. When $n = 3$, every nonempty proper subset of N has one or two agents. □

Lemma O8. *Let Ω be finite, and let μ belong to the Dirichlet family. For every profile of monotone preferences, the percentile vector $(U_1(x_1), \dots, U_n(x_n))$ is negatively associated under μ.*

Proof. We first prove negative association for the shares of a single good. Write $\mathbf{D}$ for its Dirichlet vector of agent shares and unused resources. Under the standard gamma representation, for any partition into disjoint coordinate groups A and A^c, $\mathbf{D}_A = S\mathbf{Z}$ and $\mathbf{D}_{A^c} = (1 - S)\mathbf{Y}$, where $S = \sum_{k \in A} D_k$ and $S, \mathbf{Z}, \mathbf{Y}$ are independent. As S rises, the group A receives a larger total share and the complementary group receives a smaller one. For disjoint groups A, B and coordinatewise nondecreasing g, h, conditional independence yields

$$\operatorname{Cov}(g(\mathbf{D}_A), h(\mathbf{D}_B)) = \operatorname{Cov}\big(\mathbb{E}[g(S\mathbf{Z}) \mid S], \mathbb{E}[h((1 - S)\mathbf{Y}_B) \mid S]\big) \leq 0.$$

The first conditional mean increases with S, the second decreases, and the covariance of oppositely monotone functions of one random variable is nonpositive. For example, for an independent copy S', twice this covariance is the expectation of $(g_0(S) - g_0(S'))(h_0(S) - h_0(S')) \leq 0$, where g_0, h_0 denote the two conditional means. Thus one good's Dirichlet shares are negatively associated.

The closure arguments do not require conditioning a Dirichlet vector on selected shares. If X and Y are independent negatively associated vectors, and g, h are bounded increasing functions of disjoint coordinate groups in (X, Y), condition on Y and apply negative association of X. This bounds $\mathbb{E}[gh \mid Y]$ by the product of the two conditional means. Those means are increasing functions of disjoint coordinate groups of Y, so negative association of Y gives $\mathbb{E}[gh] \leq \mathbb{E}[g]\mathbb{E}[h]$. Iterating over the finitely many independent goods proves negative association of all shares jointly. Composing increasing functions with increasing maps on disjoint coordinate groups preserves the same inequality (Joag-Dev and Proschan, 1983). Each agent's percentile $U_i(x_i)$ is nondecreasing in her own goods, and agents use disjoint groups of share coordinates. The percentile vector is therefore negatively associated. □

Proof of Corollary O2. For every law μ of the Dirichlet family, the pair law $\mu \otimes \mu$ satisfies Assumption 1. By Lemma O8 and Proposition O3, condition (S) holds, and Theorem O3 gives a balanced calibration. □

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