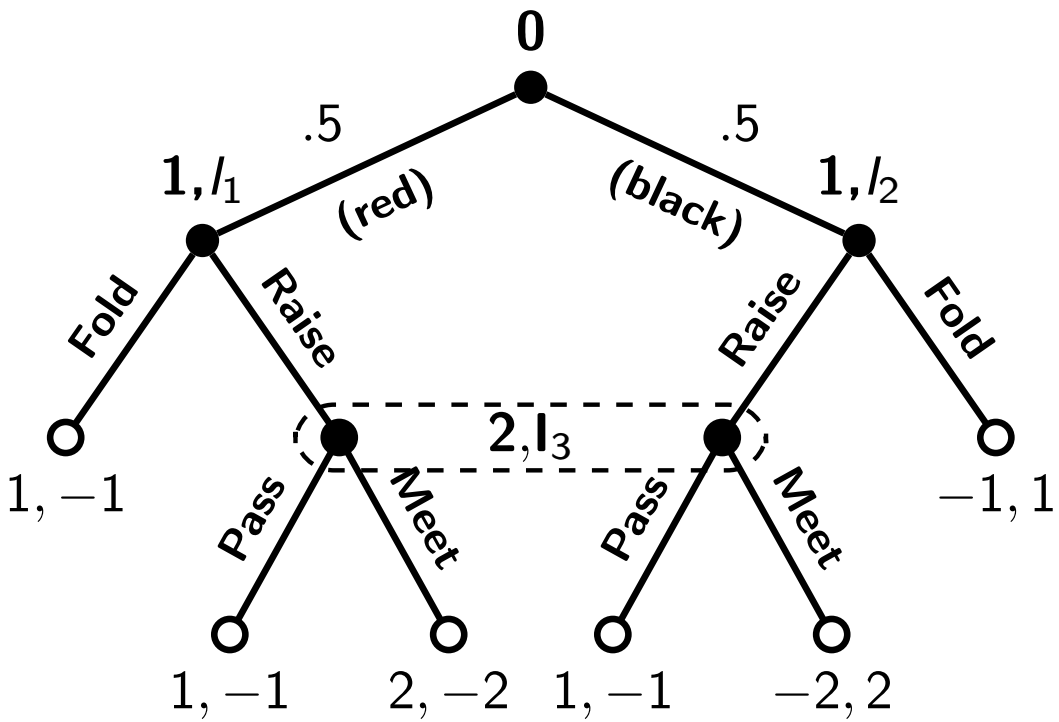
Incomplete Information in Extensive-Form Games

Summer 20

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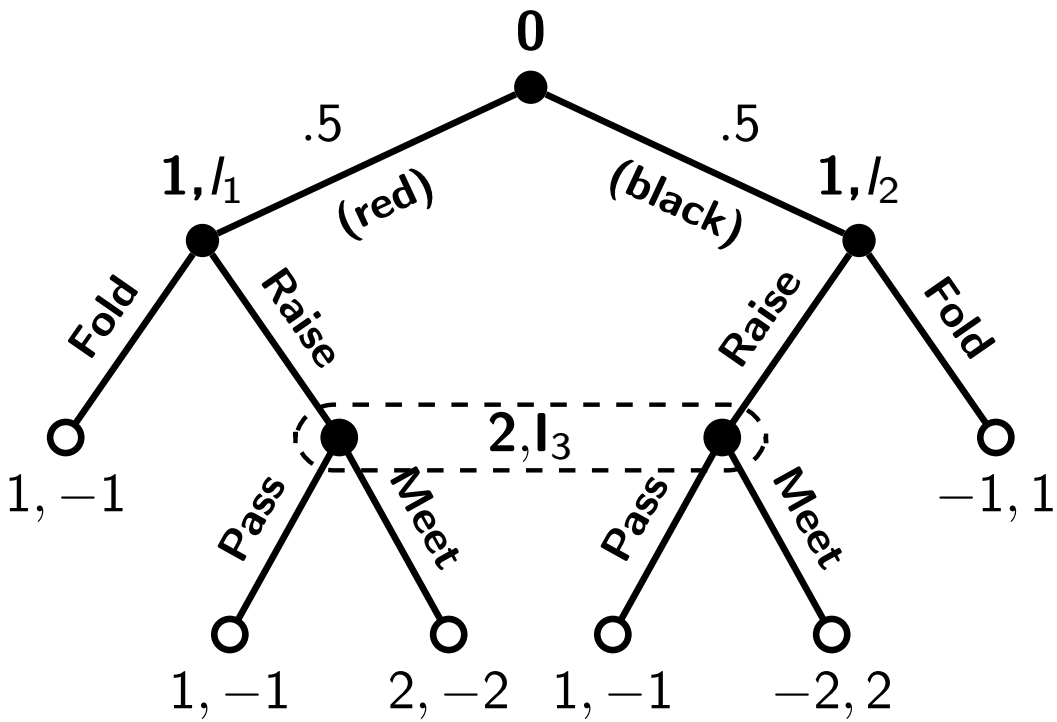
Incomplete information, information sets, PBE

- Extensive-form game Players may move simultaneously, or may not be able to observe all moves perfectly
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Subgame-perfect equilibrium (SPE)

An strategy profile σ is an SPE if it induces optimal choice (probability) for all players in all information sets.

Mixed Strategies

A mixed strategy σ_i is a function that assigns for each information a probability distribution over actions.

So, given an information set I, $\sigma_i(I)$ specifies with which probability the agent chooses each action. I.e. $\sigma_i(I)(a)$ is the probability of choosing a at the information set I.

Beliefs and Bayesian updating

At every information set, Players have beliefs about which node is the true node in which they have to make decision, i.e., they assign a probability to each node in the information set.

- E.G Player 2 assigns a probability p to the fact the Player 1 has drawn a red card and a probability $1 - p$ to the fact the Player 1 has drawn a black card.

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The Posterior beliefs specify the Players' beliefs after observing some moves.

- After observing that Player 1 decided to raise (at the information set I_3), Player 2 assigns a probability $\mu_2(red|r)$ to the fact the Player 1 has drawn a red card and a probability $1-\mu_2(red|r)$ to the fact the Player 1 has drawn a black card.
- Posterior beliefs are determined with Bayesian updating

$$\mu_2(r|I_3) = \frac{\mathbb{P}(I_3|red)\mu_0(red)}{\mathbb{P}(I_3|red)\mu_0(red) + \mathbb{P}(I_3|black)\mu_0(black)} =$$

$$= \frac{\sigma_1(red)(r)\mu_0(red)}{\sigma_1(red)(r)\mu_0(red) + \sigma_1(black)(r)\mu_0(black)}$$

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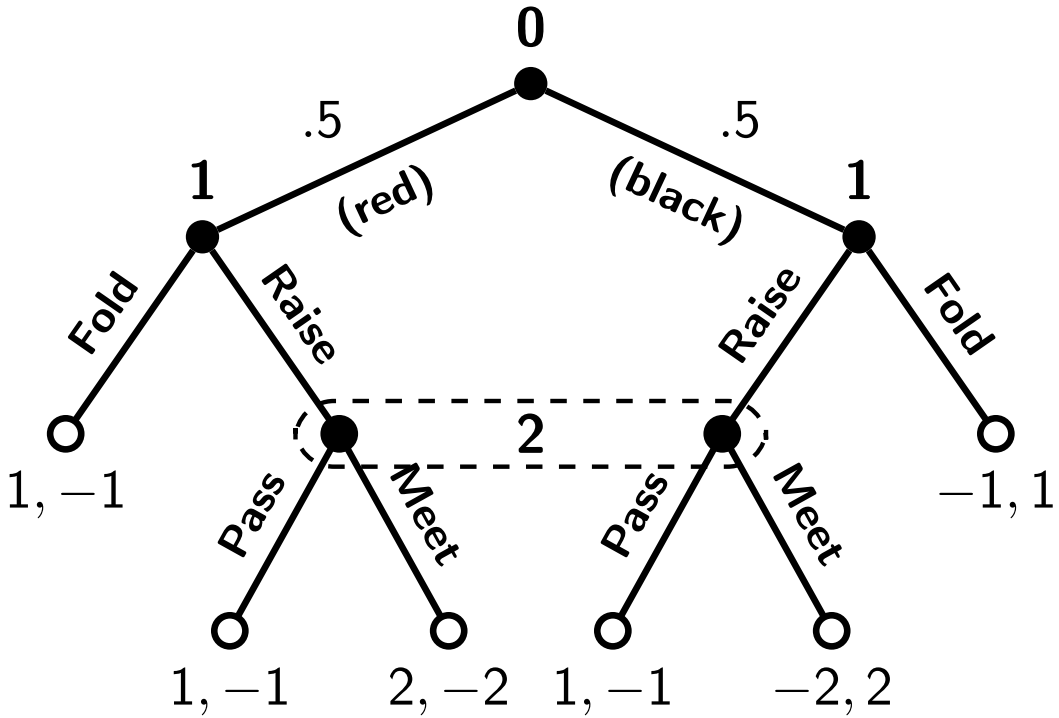
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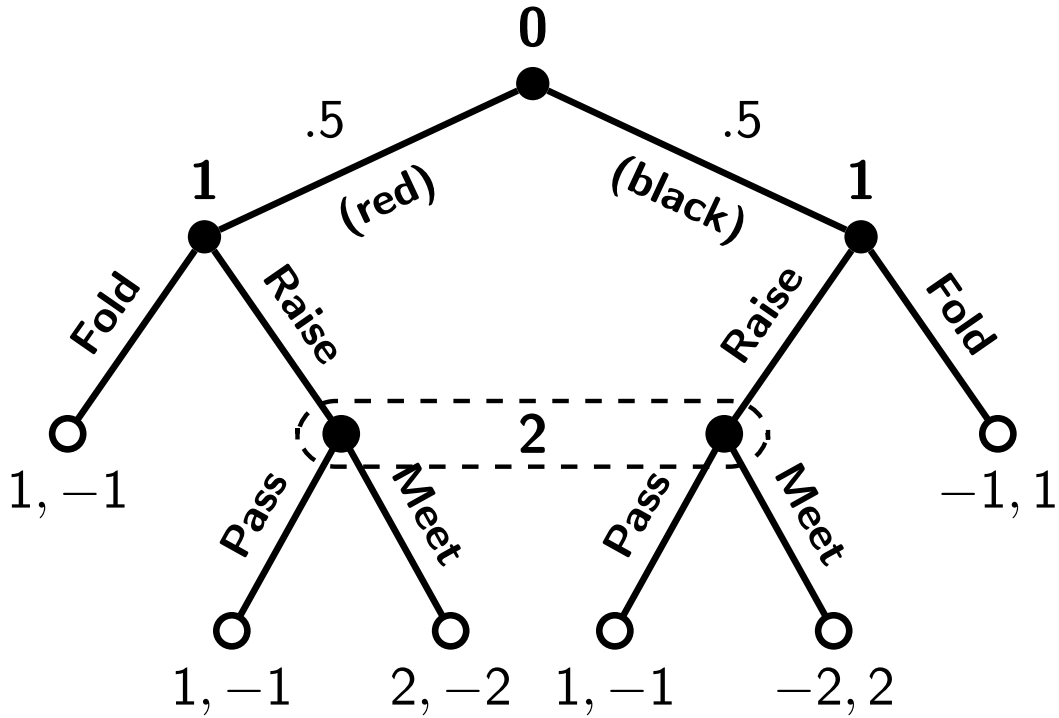
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Perfect-Bayesian Equilibrium



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Perfect-Bayesian Equilibrium (PBE)

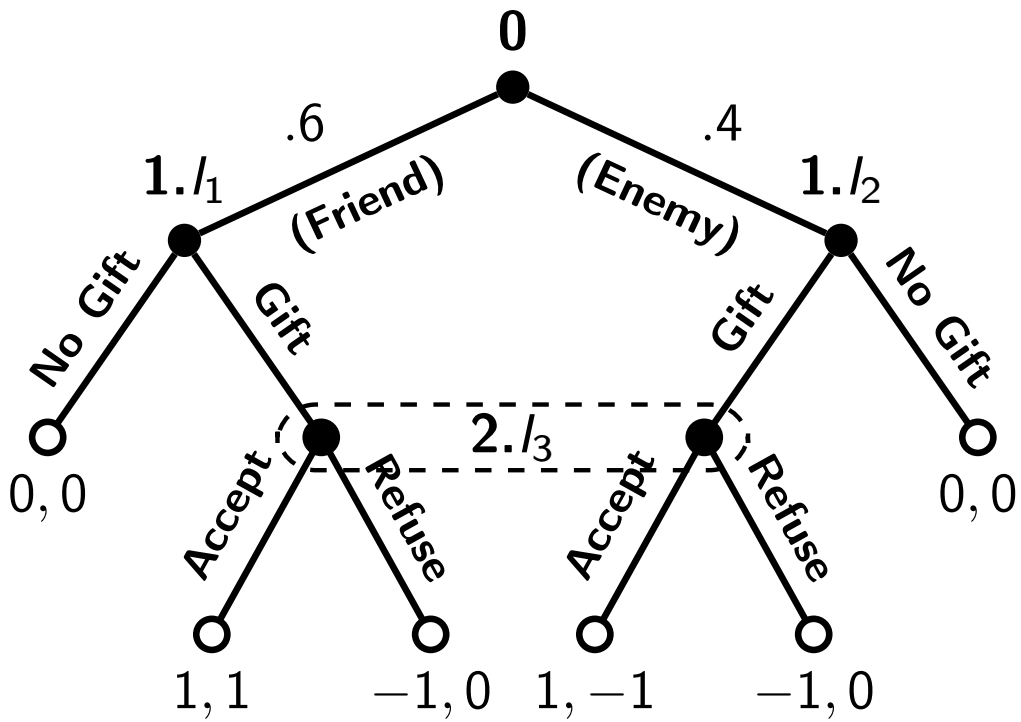
(σ, μ) is a PBE if:

- Each player holds beliefs μ over other players' unobserved moves and these beliefs are compatible with Bayes' Rule given strategy profile σ;
- σ induces optimal actions for all players in all information sets given beliefs μ.

Example

- There are two players
- Player 1 is a friend of Player 2 (with 0.6 probability) or an enemy (with 0.4 probability);
- Player 1 is the only one who knows her type. First, she decides if she wants to give a gift to Player 2 or not;
  - **If she does not** then both Players get 0;
  - **If she does** Player 2 must decide whether she accepts or refuses
    - If she refuses then Player 1 gets -1 and Player 2 gets 0;
    - If she accepts then Player 1 gets 1 and Player 2 gets 1 if Player 1 is a friend, and she gets -1 otherwise.

Obtaining Perfect-Bayesian Equilibrium



Strategies

- **Player 1:** GG, GN, NG, NN
- **Player 2:** A, R

Notations

Let μ be the posterior beliefs of players

- $\mu_1(a|I_i)$ Player 1's posterior belief that Player 2 will choose a when she is at the information set I_i $(i = 1, 2)$.
- $\mu_2(a|I_3)$ Player 2's posterior belief regarding Player 1's type at the information set I_3.

Let $\pi_i(\sigma_i|I)$ be the expected payoff of Player i when she chooses the strategy σ_i a the information set I.

Obtaining Perfect-Bayesian Equilibrium

If Player 1's strategy is GG then

- Player 2 does not learn anything from Player 1's decision and her posterior belief must equal her prior belief: $\mu_2(friend|I_3) = \mu_0(friend) = 0.6$;
- Thus, $\pi_2(A|I_3) = 0.6 - 0.4 = 0.2 > 0 = \pi_2(R|I_3)$, so she would choose to *Accept*.
- Player 1 knows it, so $\pi_1(GG|friend) = \pi_1(GG|enemy) = 1$ which is better than every other strategy whatever her type be. Therefore, GG is the best response to Player 1 choosing accept.
- Thus, (GG, A) with beliefs $(\mu_1(Accept) = 1, \mu_2(friend|I_3) = 0.6)$ is a PBE.

If Player 1's strategy is GN then

- Player 2 knows what is player 1's type: when Player 1 chooses Gift $\mu_2(friend|I_3) = 1$, and when she does not, then $\mu_2(enemy|I_3) = 0$.
- Thus, is optimal to Accept when player 1 gives a Gift.
- Player 1 knows it, thus, if she gives a gift, then believes that Player 2 is going to accept: $\mu_1(Accept) = 1$. But in that case, she would be better off choosing GG. Since when she is Player's 2 enemy she gets 1 if she give a gift since Player 2 accepts for sure, while she gets 0 if she does not.
- Thus, GN cannot be in a PBE

Obtaining Perfect-Bayesian Equilibrium

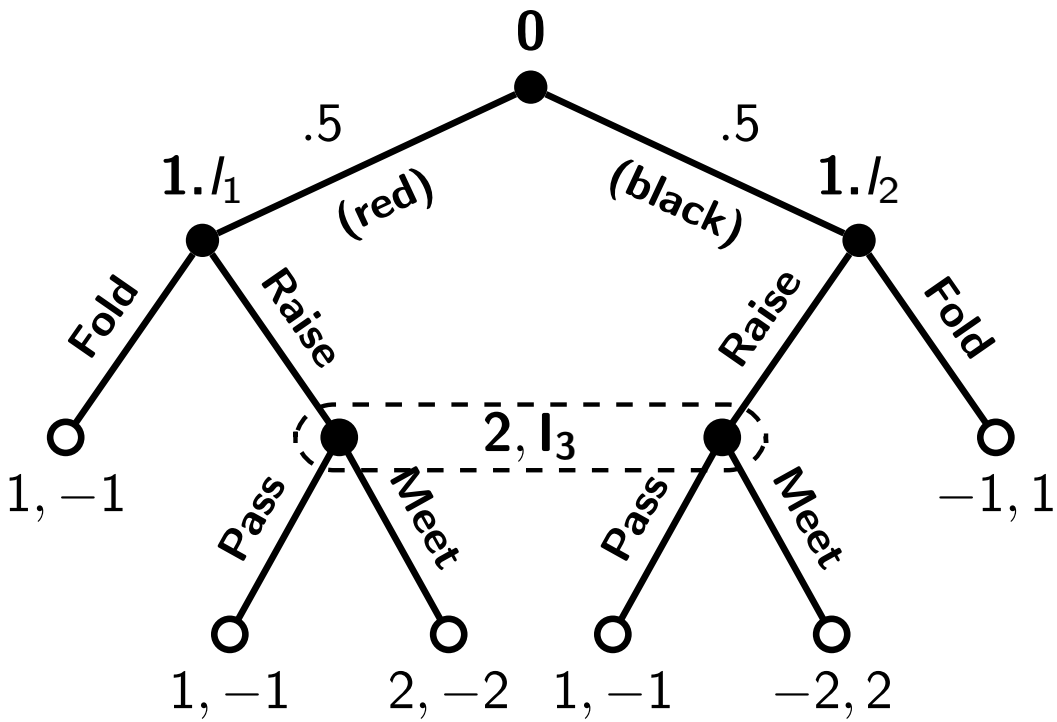
If Player 1's strategy is NN then

- Player 2 does not learn anything from Player 1's decision and her posterior belief must equal her prior belief: $\mu_2(friend|N) = 0.6$;
- Thus, $\pi_2(A|I_3) = 0.6 - 0.4 = 0.2 > 0 = \pi_2(R)$, so she would choose to *Accept*.
- Player 1 knows it, so $\mu_1(Accept) = 1$. Thus, she would be better off choosing GG again.
- Thus, there are no PBE with *NN*.

If Player 1's strategy is NG then

- Player 2 knows what is player 1's type: when Player 1 chooses Gift $\mu_2(friend|I_3) = 0$, and when she does not, then $\mu_2(enemy|I_3) = 1$.
- Thus, this is optimal to Refuse when player 1 gives a Gift.
- Player 1 knows it, thus, if she gives a gift, then she believes that Player 2 is going to refuse it: $\mu_1(Accept) = 0$. Her best response then is not to give a gift whatever at type.
- Thus, there are no PBE with *NG*.

Obtaining Perfect-Bayesian Equilibrium



- Player 1's information sets I_1="the card is red", or I_2="the card is black";
- Player 1's information sets I_3="Player 1 has chosen Raise";

Notations

Player 1's mixed strategies are the functions σ_1 such that

- $\sigma_1(I_1)(r)$ is Player 1's Pr of choosing r when she has drawn a red card,
- $\sigma_1(I_1)(f)$ is Player 1's Pr of choosing f when she has drawn a red card,
- $\sigma_1(I_2)(r)$ is Player 1's Pr of choosing r when she has drawn a black card,
- $\sigma_1(I_2)(r)$ is Player 1's Pr of choosing f when she has drawn a black card,

Player 2's mixed strategies are the functions σ_2 such that

- $\sigma_2(I_3)(m)$ is Player 2's Pr of choosing m when Player 1 has chosen r,
- $\sigma_2(I_3)(p)$ is Player 2's Pr of choosing p when Player 1 has chosen r,

Solution

The Perfect-Bayesian Equilibirum is (σ_1^*, σ_2^*) with beliefs $\mu = (\mu_1, \mu_2)$ where

- $\sigma_1^*(I_1)(r) = 1$ and $\sigma_1^*(I_3)(r) = 1/3$
- $\sigma_2^*(I_3)(m) = 2/3$
- $\mu_1(m|I_1) = \mu_1(m|I_2) = 2/3$
- $\mu_2(red|I_3) = 3/4$

Check

μ is compatible with Bayes' rule

$$\mu_2(red|I_3) = \frac{1/2\sigma_1^*(I_1)(r)}{1/2\sigma_1^*(I_1)(r) + 1/2\sigma_1^*(I_2)(r)} = \frac{3}{4}$$

σ induces optimal actions for both Players in all I :Given the beliefs μ we have

- $\pi(\sigma_1|I_1) = 2(1 + 2\sigma_1(I_1)(r) \implies \pi(\sigma_1^*|I_1) \geq \pi(\sigma_1|I_1)$ for all σ_1 Thus,

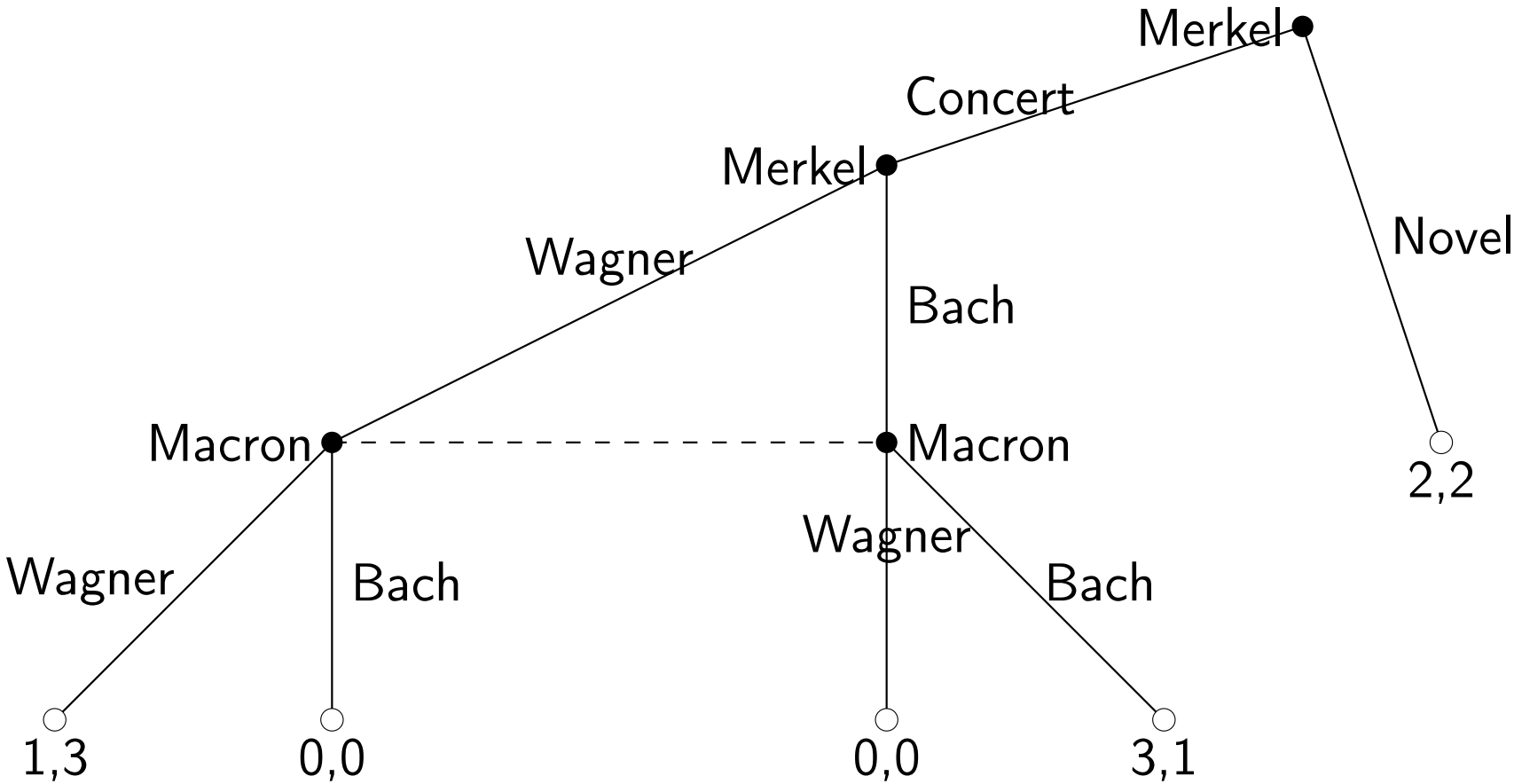
$$\mu_2(red|I_3) = \frac{1}{1 + \sigma_1^*(I_2)(r)} \implies \sigma_1^*(I_2)(r) = \frac{1 - \mu_2(red|I_3)}{\mu_2(red|I_3)}$$

- $\pi(\sigma_1|I_2) = \sigma_1(I_2)(r)(2 - 3\mu_1(m|I_2)) - 1 = -1 \implies \pi(\sigma_1^*|I_2) \geq \pi(\sigma_1|I_2)$ for all σ_1
- $\pi(\sigma_2|I_3) = \sigma_2^*(I_3)(m)(3 - 4\mu_2(red|I_3)) - 1 \implies \pi(\sigma_3^*|I_3) \geq \pi(\sigma_3|I_3)$ for all σ_2

Backward Induction vs Forward Induction

- **Backward Induction:** Agents believe that all future plays will be rational.
- **Forward Induction:** Agents believe that all past plays were rational.

Example:


Merkel
Concert
Merkel
Novel
Wagner
Bach
Macron
Macron
2,2
Wagner
Wagner
Bach
Bach
1,3
0,0
0,0
3,1

Problem

Subgame 1:

	W	B
CW	$3,1$	$0,0$
CB	$0,0$	$1,3$
NW	$2,2$	$2,2$
NB	$2,2$	$2,2$

Subgame 2:

	W	B
CW	$3,1$	$0,0$
CB	$0,0$	$1,3$

The Subgame-Perfect Equilibria are (CW, W) and (NB, B). Are they both reasonable?

- Imagine that Macron finds himself playing to the second subgame. Is it reasonable to believe that Merkel has chosen Bach? Probably not. Merkel could have guaranteed herself 2 by playing N. Why would she enter a subgame and play in a manner in which she is only going to get 1?
- Arguably (NB, B) is not reasonable, despite being a subgame Perfect-equilibrium.

This is an example of forward induction reasoning.

Central to the Forward Induction concept is

- that previous play tells you something about future play
- Subgames cannot be treated in isolation

Thus, forward Induction is not a refinement of SPNE.

Despite it's plausibility forward induction has proved tricky to formalize. So, it is beyon the scope of the course.