Decisions Under Temporal Resolution of Uncertainty

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January 2022

Examples

- Intertemporal consumption with uncertain income;
- Investing in assets with evolving returns;
- Precautionary savings.

Risk through time

Up until now, we have been considering decisions with,

	Certain	Risky	Uncertain
Static	$U(x) = u(x)$	$U(l) = \mathbb{E}_l u(.)$	$U(f) = \mathbb{E}_\mu u(f(.))$
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- For all lotteries $l \in \triangle(Z^T)$.

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- For all lotteries $l \in \triangle(Z^T)$.
- This gives us the **Expected Discounted Utility Model (EDU)**.

Drawbacks

This model is simple and elegant but we may suspect that there are some issues.

Deep reason: When you combine **Risk** and **Time** the agent is

- uncertain about the period 2;
- uncertain about the uncertainty she will face in period 2 about period 3, and so on...

Cannot be capture by **EDU** because it is based on $\triangle(Z^T)$.
Example: The lotteries $l = \frac{1}{2} \cdot \delta_{(5,0)} + \frac{1}{2} \cdot \delta_{(5,10)}$ and $l' = \delta_{(5, \frac{1}{2} \cdot \delta_0 + \frac{1}{2} \cdot \delta_{10})}$ are the same here.

EDU and correlation

Consider the assets represented by the correlated lottery $l = 1/2 \cdot \delta_{(1,1,...,1)} + 1/2 \cdot \delta_{(0,0,...,0)}$ and the uncorrelated lottery $l' = \left(1/2 \cdot \delta_0 + 1/2 \cdot \delta_1, 1/2 \cdot \delta_0 + 1/2 \cdot \delta_1, ..., 1/2 \cdot \delta_0 + 1/2 \cdot \delta_1\right)$.

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Then, according to EDU we have

$$U(l) = \frac{1}{2}\sum_{t=1}^{T} \delta^t u(1) + \frac{1}{2}\sum_{t=1}^{T} \delta^t u(0) = \sum_{t=1}^{T} \delta^t (\frac{1}{2}u(1) + \frac{1}{2}u(0)) = U(l')$$

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- But the agent may be willing to "hedge" consumption risk over time and be "correlation averse." (e.g. *aversion to long-run consumption risk* see (Duffie and Epstein, 1992))
- Or she may be "correlation-seeking."

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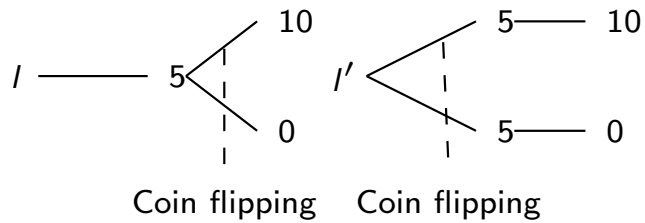
- **Intertemporal substitution:** what amount of *current consumption* should be sacrificed for *future consumption*.
- **Risk aversion:** the extent to which risk should be avoided.

Problem: with EDU, both factors are captured by u', and cannot be disentangled but

- These are different concepts psychologically speaking;
- Empirically we would like to separate them;
- Can lead to puzzles (e.g. the *equity premium puzzle*).

EDU implies indifference to the timing of the resolution of uncertainty

A fair coin is flipped. If heads, then $(z_0, z_1) = (5, 10)$; if tails, $(z_0, z_1) = (5, 0)$.
The timing of the flipping might be relevant, though not distinguished EDU.

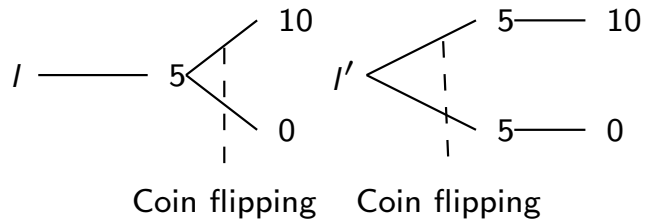


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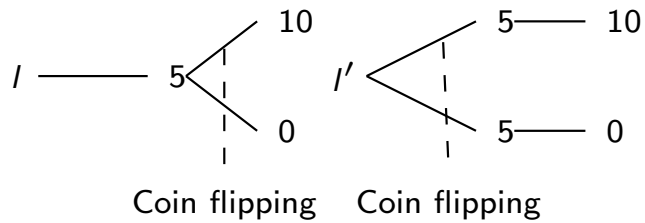
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- Preference for late resolution$= l \succ l' =$ hope, suspense, ...

Issues with EDU

Expected Discounted Utility is simple and elegant, but it seems that we are loosing relevant information with it.

- EDU implies indifference to correlation:
- EDU does not allows for disentangling the effect of intertemporal substitution and the effect of risk aversion;
- EDU implies indifference to the timing of the resolution of uncertainty.

Kreps and Porteus's model of temporal resolution of uncertainty

Temporal lotteries

EDU is based on von Neumann-Morgenstern lotteries (i.e. elements of $\triangle(Z^T)$), which do not encode when uncertainty resolves.

To capture preference for temporal resolution of uncertainty we need lotteries that encode the timing of resolution: *Temporal lotteries.*

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- Third, define $Y_1 = Z_1 \times L_2$ and L_1 the set of lotteries over Y_1.

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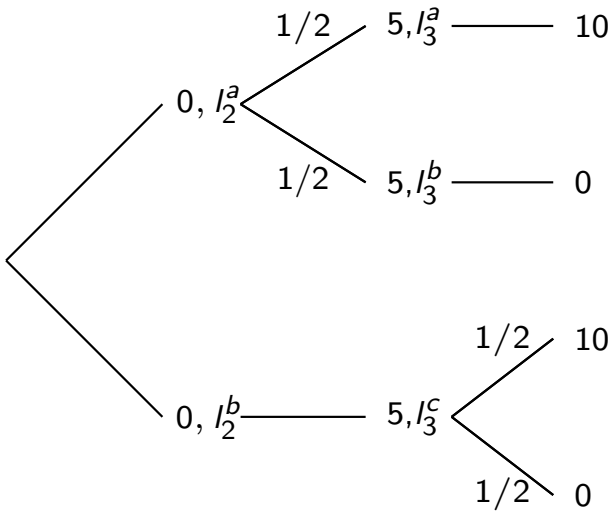
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Definition

Elements of L_1 are called *temporal lotteries*.

Temporal lotteries and decision trees



$$l_1^a \equiv \left(\delta_0, l_2^a\right)$$

$$= \left(\delta_0, \left(1/2(\delta_5, l_3^a) + 1/2(\delta_5, l_3^b)\right)\right)$$

$$= \left(\delta_0, \left(1/2(\delta_5, \delta_{10}) + 1/2(\delta_5, \delta_0)\right)\right)$$

$$l_1^b \equiv \left(\delta_0, l_2^b\right) = \left(\delta_0, \left(\delta_5, l_3^c\right)\right)$$

$$= \left(\delta_0, \left(\delta_5, 1/2\delta_{10} + 1/2\delta_0\right)\right)$$

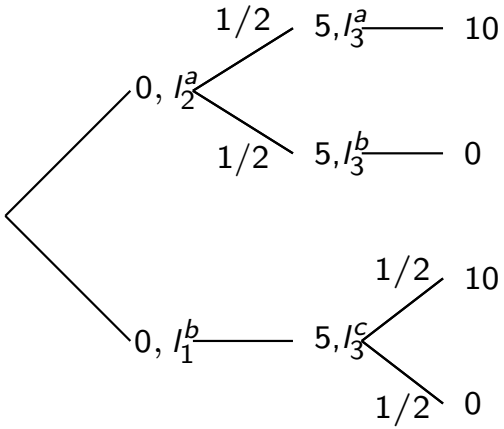
Recursive Utility Function

Let $W_t : Z \times \mathbb{R} \mapsto \mathbb{R}$ and $U_T : Z \mapsto \mathbb{R}$ and

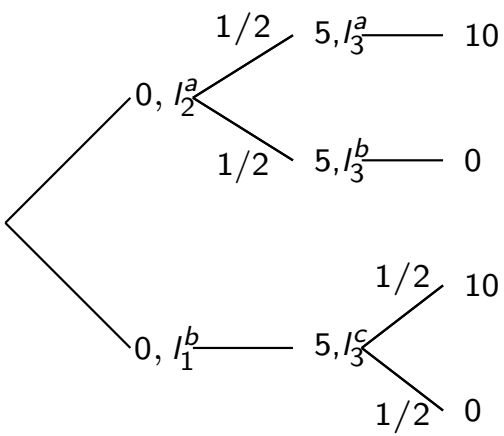
$$U_t(z_t, I_{t+1}) = W_t\left(z_t, \mathbb{E}_{I_{t+1}} U_{t+1}(.)\right)$$

with each W_t bounded, continuous, and **increasing** in its second argument.

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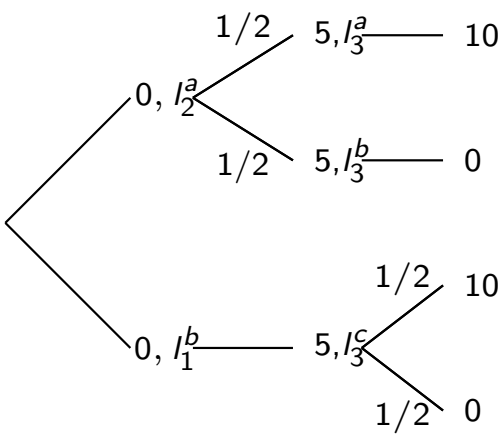
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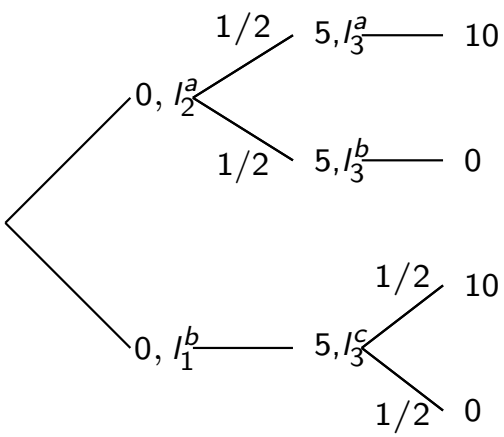
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Second, we define U_2

- $U_2(5, I_3^a) = W(5, U_3(10)) = W(5, 8)$
- $U_2(5, I_3^b) = W(5, U_3(0)) = W(5, 0)$
- $U_2(5, I_3^c) = W(5, \frac{1}{2}U_3(0) + \frac{1}{2}U_3(10)) = W(5, \frac{1}{2}\underline{u} + \frac{1}{2}\bar{u})$

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- $U_1(0, I_2^a)) = W\left(0, \mathbb{E}_{I_2^a}(U_2(.))\right) = W\left(0, \frac{1}{2}U_2(5, I_3^a) + \frac{1}{2}U_2(5, I_3^b))\right) = W\left(0, \frac{1}{2}W(5, \bar{u}) + \frac{1}{2}W(5, \underline{u}))\right)$
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Recursive Utility Function

For each t, let $W_t : Z \times \mathbb{R} \mapsto \mathbb{R}$ and define $U^T : Z \mapsto \mathbb{R}$ and

$$U_t(z_t, l_{t+1}) = W_t\left(z_t, \mathbb{E}_{l_{t+1}} U^{t+1}(.)\right) \tag{1}$$

with each W_t bounded and continuous **increasing** in its second argument.

Definition

The agent has Kreps-Porteus preferences $\succeq$ if there exists a sequence $(W_t)_t$ and $(U_t)_t$ defined as in (1)

$$l_1 \succeq l_1' \iff \mathbb{E}_{l_1}(U_1) \geq \mathbb{E}_{l_1'}(U_1)$$

Interpreting the aggregator W

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The function W_t is called the **aggregator**: at each period it *aggregates* the utility over *current consumption* and the utility over *future consumption*.

Examples:

- **EDU:** $W(z, x) = u(z) + \delta x$, then $\mathbb{E}_I U(.) = \mathbb{E}_I \left[\sum_{t=1}^{T} \delta^t u(.) \right]$
- **Ursawa preferences:** $W(z, x) = u(z) + \delta(u(z))x$
- **Epstein-Zin preferences:** $W(z, x) = (u(z)^\rho + \delta \cdot x^{\frac{\rho}{\alpha}})^{\frac{1}{\rho}}$ Epstein and Zin (1989)).

A necessary and sufficient condition on W for preference for early resolution of uncertainty

Clearly,when the DM prefers early resolution of uncertainty we have $l_1^a \succ l_1^b$ if and only if

$$W\left(0, \frac{1}{2}W(5,\bar{u}) + \frac{1}{2}W(5,\underline{u})\right) > W\left(0, W\left(5, \frac{1}{2}\underline{u} + \frac{1}{2}\bar{u}\right)\right)$$

$$\iff \frac{1}{2}W(5,\bar{u}) + \frac{1}{2}W(5,\underline{u}) > W\left(5, \frac{1}{2}\underline{u} + \frac{1}{2}\bar{u}\right)$$

which is true whenever W is convex in its second argument.

Theorem[Kreps and Porteus (1978)] *Consider a DM with a Kreps-Porteus preference $\succ$ and its associated aggregator W.*

- *If $W(.,x)$ is convex on x, then the DM has a preference for early resolution of uncertainty.*
- If $W(.,x)$ is concave on x, then the DM has a preference for late resolution of uncertainty.
- If $W(.,x)$ is linear on x, then the DM is indifferent to the timing of resolution of uncertainty.

Epstein-Zin preferences

EDU cannot disentangle risk aversion and intertemporal preference (both captured by u'). Yet, it can be done with recursive utility and notably Epstein-Zin preferences.

Epstein and Zin (1989) propose a recursive utility model with an aggregator

$$U_t(z_t, I_{t+1}) = \left(z^\rho + \delta \left(\mathbb{E}_{I_{t+1}} U_{t+1}^\alpha(.) \right)^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}}$$

- Special case of Kreps and Porteus (1978);
- $\frac{1}{1-\rho}$ is the elasticity of intertemporal substitution;
- $1 - \alpha$ is the coefficient of risk aversion.

Epstein-Zin preferences and the Equity Premium Puzzle

Equity premium puzzle: To explain the empirically observed Difference of expected return between stock (risky assets) and bond (risk free assets), we need to assume a an implausibly high risk aversion rate.

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Partial explanation: Breaking the ties between elasticity of consumption and risk aversion (Bansal and Yaron (2004)) by using Epstein-Zin preferences. Can explain only a third of the equity premium...

Axioms and Representation

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Axiom 1: $\succeq$ is transitive and complete.

Axiom 2: $\succeq$ is continuous.

Axioms

Let $P_t(z_1, ...z_t) \subseteq L_1$ the temporal lotteries such that $z_1, ...z_t$ happens with certainty.

- $l_1^a \in P_1(0)$ but $l_1^a \notin P_2(0,5)$ or
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Axiom 3: For any t, if $l_1, l_1' \in P_t(z_1, ..., z_t)$ satisfy $l_1 \succ l_1'$, then $\alpha l_1 + (1-\alpha) l_1'' \succ \alpha l_1 + (1-\alpha) l_1''$ for all $l_1'' \in P_t(z_1, ..., z_t)$

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Axiom 4: For any t, and any $l_1^1, l_1^2 \in P_t(z_1, ..., z_t)$ and $l_2, l_2' \in P_t(z_1', ..., z_t')$, such that l_1^1 differs from l_1^3 only before period t and l_1^2 differs from l_1^4 only before period t, then

$$l_1^1 \succeq l_1^2 \iff l_1^3 \succeq l_2^4$$

The representation

Theorem:[Kreps and Porteus (1978)] A preference is a Kreps Porteus preference if and only if it satisfies A1-A4.

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- For any $z_1 \in Z$ and any $l_1 = (z_1, l_2), l_1' = (z_1, l_2) \in P_1(z_1)$, $\exists U_2^{z_1}$ such that $l_1 \succeq l_1' \iff \mathbb{E}_{l_1} U_2(z_1, .) \geq \mathbb{E}_{l_1'} U_2(z_1, .)$.

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Hence, given a prize z_1, both expected utilities represent the same preference (this results from *temporal consistency* see below).

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We have built W_1 and U_1 that relates period 1 to period 2. Others W_t and U_t can be built recursively with a similar reasoning but they will depend on the past history $z_1, ..., z_{t-1}$.

Proof of sufficiency

So, we obtain that there exist functions $W^{z_1,\dots,z_{t-1}} : Z \times \mathbb{R} \mapsto \mathbb{R}$ and $U_t^{z_1,\dots,z_{t-1}} : Z \times \mathbb{R} \mapsto \mathbb{R}$ representing our preference and such that

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with each $W_t^{z_1,\dots,z_{t-1}}$ and bounded and continuous **increasing** in its second argument.
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What left to be shown is that W_t and U_t do not depend on $z_1, \dots, z_{t-1}$. To show that take two lotteries such that $l_1 \in P^t(z_1, \dots z_t)$ and $l_1' \in P^t(z_1', \dots z_t')$ and which differs only before t. Then, by A4 we have

$$U_t^{z_1,\dots,z_{t-1}}(z_t, l_t) \underbrace{=}_{A4} U_t^{z_1',\dots,z_{t-1}'}(z_t, l_t) \underbrace{=}_{Def} U_t^{z_1',\dots,z_{t-1}'}(z_t, l_t')$$

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2. we could have easily incorporated uncertainty over the menus that will be faced in the future and dealt with the situation above. Kreps and Porteus (1978) do it, we did not for the sake of simplicity.

Temporal resolution of Uncertainty with non-expected utility

Non-Expected Utility as a Certainty Equivalent

One could imagine an even more general type of recursive preference such as

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Non-Expected Utility and attitudes towards the timing of uncertainty

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Bad news: this means that in non-expected utility models we cannot disentangle risk attitude (captured by $\mathcal{I}$) from attitudes towards the timing of uncertainty (captured by W).

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Answer: Yes, when $\mathcal{I}$ is the maxmin operator but no for others ambiguity models (Strzalecki (2013)).

Timing of Uncertainty Resolution and Anticipatory feelings

Psychological Expected Utility

"Worry is a cognitive phenomenon, it is concerned with future events where there is **uncertainty** about the outcome, the future being thought about is a negative one, and this is accompanied by feelings of **anxiety**" (MacLeod et al., 1991, p.478).

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Examples:

- providing information prior to a medical procedure would stimulate the "work of worrying" that would initially raise anxiety, but subsequently lower anxiety (and possibly speed recovery) (Janis, 2016).
- subjects who had earlier given a blood sample were told that their blood sample had been analyzed to identify whether or not they were carriers of one of two genetic mutations that indicate susceptibility to breast cancer later in life. Subjects were then asked whether or not they wanted to know the result of the test. A large set of experimental subjects reject this information. (Lerman et al., 1998)

Three components

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In the model there are two periods $t = 1, 2$ and three main component.

1. A formal description of the space of lotteries in the physical world (temporal lotteries);
2. A space of psychological states (such as anxiety, excitement,...) with preferences on it;
3. A mapping that connects physical lotteries with psychological states.

Physical space

The physical space is simply a set of *temporal lotteries* L_1, i.e.

- Z_t=prize at t;
- L_2=lotteries over period 2 prize;
- $Y_1 = Z_1 \times L_2$;
- L_1=lotteries over Y_1, i.e. *temporal lotteries*.

The space of psychological states

- In the first period there is a set the possible psychological states
- Let $X = X_1 \times Z_2$;
- $P(X) =$ the set of probability distribution on X;
- $\succeq =$ preference on $P(X)$.

Assumptions: $\succeq$ is such that (i.e. it satisfies some axioms) there exists a continuous function $U : X \mapsto \mathbb{R}$ such that for $p, q \in P(X)$, $p \succeq q$ if and only if $\mathbb{E}_p U \geq \mathbb{E}_q U$ and

$$U(x) = u_1(x_1) + u_2(z_2)$$

where $u_t : X_t \mapsto \mathbb{R}$.

Connecting physical prizes with psychological states

Let $\phi : Y_1 \mapsto X_1 =$ gives the psychological states that results from the agent facing $y_1 = (z_1, l_2) \in Y_1$.

The **utility function over temporal lotteries** is induced by ϕ and the utility function over psychological states, i.e.

$$V_1(z_1, l_2) = u_1(\phi(z_1, l_2)) + \mathbb{E}_{l_2}[u_2(z_2)]$$

Hence, we can define $\succsim$ for the temporal lotteries $l_1, l_1' \in L_1$

$$l_1 \succsim l_1' \iff \mathbb{E}_{l_1} V_1(y_1) \geq \mathbb{E}_{l_1'} V_1(y_1)$$

States, menus, and strategies

There are state $s_i \in S_i$ for each period.

At period 1, the agent

- chooses an action in $\alpha_1 \in \Gamma_1(s_1) \subseteq A_1$...
- ... and obtains a first-period payoff according to $\nu : A_1 \mapsto Z_1$

The state changes from s_1 to s_2 with a probability of density $Q(s_2|s_1)$.

At period 2, the agent chooses a lottery in $\Gamma_2(s_2, \alpha) \subseteq L_2$.

Definition

A strategy is a tuple (α_1, π_2) where 1) α_1 is a first-period action, 2) $\pi_2 : A_1 \times S_2 \mapsto L_2$ is a second-period action with $\pi_2(\alpha_1, s_2) \in \Gamma_2(s_2, \alpha_1)$. The set of every such π_2 is Π.

The decision timeline

Time Inconsistency

The lottery over second-period prizes that the agent anticipates at the end of period 1 is

$$\lambda(\alpha_1, \pi_2|s_1) = \int \pi(\alpha_1, s_2) Q(s_2|s_1) ds_2$$

This means that the first-period temporal lotteries (i.e lotteris over $(z, \lambda(\alpha_1, \pi_2|s_1)))$ depends on the second-period strategyπ_2.

- This gives rise to time inconsistency of optimal choices;
- She chooses (α_1, π_2) in the first period and $\pi' \neq \pi$ in the second period (because she is anxious about her future feelings in the first period);
- Note that was ruled out by the temporal consistency assumption in (Kreps and Porteus, 1978).

Consistent planning and Strategy

Given (α_1, s_2) the second-period optimal strategies are in

$$G_2(\alpha_1, s_2) = \left\{ l_2 \in \Gamma(\alpha_1, s_2) | \mathbb{E}_{l_2} u_2(z_2) = \max_{l_2' \in \Gamma_2(\alpha_1, s_2)} \mathbb{E}_{l_2'} u_2(z_2) \right\}$$

An optimal strategy in period 2 belongs to

$$\Pi^\star = \{\pi_2 : A_1 \times S_2 \mapsto L_2 : \pi(\alpha_1, s_2) \in G_2(\alpha_1, s_2)\}$$

With *strategic planning* (cf. Strotz (1955)) the DM is choosing (α_1, π_2) so as to solve

$$\sup_{\substack{\alpha_1 \in \Gamma_1(s_1) \\ \pi_2 \in \Pi^\star}} u_1 \Big(\phi\big(\nu(\alpha_1), \lambda(\alpha_1, \pi_2 | s_1)\big) \Big) + \mathbb{E}_{\lambda_1(\alpha_1, \pi_2 | s_1)}[u_2(z_2)]$$

Existence

Proposition: An optimal strategy exists

Sketch of proof:

- If we consider the DM in period 1 and in period 2 as two separate individuals, our solution concept is equivalent to selecting the subgame perfect equilibrium that is best from the period 1 perspective.
- Using Harris (1985), it is easy to show that the set of such equilibrium is compact and that an optimal solution exists.

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