BEHAVIORAL ECONOMICS · LECTURE 10

Expected utility theory

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Before we start: how much would you pay?

The game

A fair coin is tossed until it first lands **heads**.

- Heads on the 1st toss: you get \$2.
- Heads on the 2nd toss: you get \$4.
- Heads on the 3rd toss: you get \$8, ... and so on: $\$2^k$ if the first head comes on toss k.

Question

What is the *maximum* price you would pay to play this game once?

Write your number down.

Learning objectives

By the end of this lecture you should be able to:

- Distinguish **risk** from **(Knightian) uncertainty**, and the main views of what a probability *is*.
- Define lotteries, expected utility, **risk aversion** and the **certainty equivalent**, and compute them.
- State the **vNM axioms** (weak order, Independence, Continuity) and show what Independence implies.
- Reproduce the main steps of the proof of the **expected utility theorem**.
- Explain why vNM utilities are unique **up to positive affine transformations**.
- Describe how Savage and Anscombe–Aumann extend expected utility to **subjective** probabilities.

Roadmap

Risk, uncertainty and the meaning of probability

Choice under risk: from expected value to expected utility

The axioms of expected utility

Choice under uncertainty

Where we are in the course

The assumptions of the Robinson model

0. **Consistent preferences:** Robinson's choices can be described by a utility function u
1. **Costless optimization:** he always finds the option that maximizes u
2. **Correct beliefs & expected utility:** he knows p and maximizes expected utility
3. **Absolute, self-regarding consumption:** only his own (b, y) enters u
4. **Stable tastes:** the same u in both periods

This lecture states **precisely** what Assumption 2 says: which axioms on choices are equivalent to "maximizing expected utility". The next lectures test them: Independence (Lecture 11), known probabilities (Lecture 12), correct beliefs (Lecture 13).

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Risk, uncertainty and the meaning of probability

Probabilities can be read as frequencies, logical relations or degrees of belief; decision theory takes the subjective route and reveals beliefs from choices.

What is ontology?

Ontology

Ontology is the study of being: what exists, and what it means to be real.

- Asks: *What kinds of things exist? How are they related?*
- Classic topics: objects, properties, events, time, causation.

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- **Realism:** reality exists independently of minds, beliefs or language.

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Two ontological positions

- **Realism:** reality exists independently of minds, beliefs or language.
- **Irrealism:** reality is mind-dependent, shaped by concepts, practices or perspectives.
- **Application:** do *probabilities* exist, or are they constructs of our mind? How are they related to the other entities of the world?

Why does ontology matter?

Any view of the world rests on an (implicit) ontology.

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The birth of modern science

Modern science rests on ontological switches:

- *Scientific revolution (Copernicus, Galileo):* truth is not found in what the divine put in us, but by observation and experiment, hence **statistics**.
- *Necessity vs. contingency:* medieval thought stressed fate and divine will; modern thought leaves room for **chance** (Pascal and Fermat, Pascal's wager).

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Anthropology

Cultures differ in their answers to ontological questions:

- Which beings have an interior life (spirits, ancestors, animals)?
- Are uncertain events due to **chance**, spirits, fate or humans?
- Can uncertainty be reduced by calculation, ritual or morality?

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Making ontological assumptions explicit helps to understand what a theory takes for granted.

Three views of probability

Frequentist

Probability = long-run relative frequency of an event when an experiment is repeated under identical conditions.

Logical

Probability = how strongly a conclusion **objectively** follows from given premises, according to rational rules of inference.

Subjective

Probability = how strongly a particular agent **believes** that an event will occur, given her information, judgment and perspective.

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The debate between these views shaped

- **Decision theory:** subjective probability.
- **Statistics:** Bayesian vs. frequentist inference.
- **Macroeconomics:** radical uncertainty (Keynes).

Probabilities as frequencies

Foundations

Jacob Bernoulli's law of large numbers (1713); the frequency interpretation of Venn and von Mises.

Idea

Probability = number of realizations of the event / number of trials, in the limit.

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Advantages

- Natural and easy to operationalize.
- Objective (?)

Critiques

- Not general: meaningless without repeated data (the next election?).
- Does not remove uncertainty: "$p \neq f$ in my sample" says nothing about p.
- No room for Bayesian conditioning: how should the statistician revise her estimate?
- Not fully objective: the statistician still chooses among tests.

Probabilities as logical relations

Foundations

Keynes (1921); Wittgenstein (1922).

Idea: probability theory is part of logic

- The degree of belief it is *rational* to hold given the evidence, not the actual (possibly irrational) beliefs of a particular person.
- Probability = an extension of propositional logic: "h makes c probable to degree α".

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Advantages

- Objective and general.
- Close to the everyday (not the mathematical) use of the word.

Critiques

- *Ontological:* "probability does not exist" (de Finetti); Ramsey: I do not perceive these logical relations (Ramsey, 1931).
- *Operational:* such probabilities are not measurable, hence hard to model.

Keynes: radical uncertainty

Keynes draws the consequences of the logical view for economics

Keynes (1937):

- "The sense in which I am using the term [uncertain] is that in which the prospect of a European war is uncertain [. . .]. About these matters there is **no scientific basis on which to form any calculable probability whatever**. We simply do not know."
- An expert who says "a war in Europe is likely" does *not* mean "there is a number α, and it is high".

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At the heart of his macroeconomics

Keynes (1936, ch. 12):

- It can be irrational to compute probabilities, or to model people as if they did.
- "I accuse the classical economic theory of being itself one of these pretty, polite techniques which tries to deal with the present by abstracting from the fact that we know very little about the future." (Keynes, 1937)
- **Animal spirits:** "a spontaneous urge to action rather than inaction", not the outcome of a weighted average of benefits times probabilities.

Subjective probability

Foundations

Ramsey (1931), de Finetti (1937), Savage (1954).

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Probability expresses an agent's **degree of belief**.

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Advantages

- General: applies to one-off events (elections, wars, start-ups).
- Requires only a weak ontology (Occam's razor): no probabilities "out there".

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Critique

Operational: how is this probability determined? Is it idiosyncratic? Does everyone have a different one?

Answer of decision theory: we **reveal** it from choices (next slides).

Subjective probability and pragmatism

Ramsey (1927)

"The essence of pragmatism I take to be this, that the meaning of a sentence is to be defined by reference to the actions to which asserting it would lead, or, more vaguely still, by its possible causes and effects."

$\Rightarrow$ The meaning of a probability lies in its **consequences for behavior**.

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Probabilities, pragmatically

- **Beliefs as guides to action:** a degree of belief is justified by the actions it supports.

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- **Revisable knowledge:** Bayesian updating mirrors inquiry as ongoing, fallible and adaptive.

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- **Beliefs as guides to action:** a degree of belief is justified by the actions it supports.
- **Anti-metaphysical stance:** no claim that probabilities exist in nature; they are tools to cope with uncertainty.
- **Revisable knowledge:** Bayesian updating mirrors inquiry as ongoing, fallible and adaptive.
- **Agent-centered:** probabilities depend on the agent's information and aims.

Risk and uncertainty

Two ways in which the future may be unknown: **roulette wheel** vs. **horse race** (Knight, 1921).

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Risk (roulette wheel)

Probabilities are **known**.

- Objective: everyone agrees on the probability of black.
- So the researcher can treat them as observable.
- Compatible with the *frequentist* view.

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(Knightian) uncertainty (horse race)

Probabilities are unknown and **subjective**.

- People may disagree about which horse will win.
- We cannot directly observe a person's beliefs.

Economics as subjectivist

Expected utility needs **two** unobservables: a utility u and a probability. How to reveal them from choices?

Revealing a belief, given u

Ramsey (1931), de Finetti (1937): compare the bet b_E = [€1 if E, €0 otherwise] with x euros for sure. Find the x at which you are indifferent. With linear u: $\mu(E) = x$.

Revealing u, given probabilities

von Neumann and Morgenstern (1947): choices between **lotteries** with known probabilities.
$\rightarrow$ Sections 2–3.

Revealing both

Savage (1954); Anscombe and Aumann (1963) $\rightarrow$ Section 4.

Recap: risk and uncertainty

Recap

- Probabilities can be read as **frequencies**, **logical relations** or **degrees of belief**.
- **Risk:** known (objective) probabilities. **Uncertainty:** unknown, subjective ones.
- Decision theory takes the **subjective, pragmatist** route: beliefs are what choices reveal.

To “reveal” beliefs from choices, we first need a theory of how people evaluate risky prospects: **expected utility**.

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Choice under risk: from expected value to expected utility

Expected value fails the St. Petersburg test; expected utility with a concave utility of money resolves it and captures risk aversion.

Lotteries

Under **risk**, the decision maker chooses between **lotteries**: probability distributions over prizes.

Example

- Four prizes: $X = \{a, b, c, d\}$ = apple, banana, celery, dragonfruit.
- The lottery $p = (0.15,\ 0.35,\ 0.5,\ 0)$: 15% chance of a, 35% of b, 50% of c, 0% of d.

Definition

Given a finite set of prizes X, a **lottery** $p \in \Delta(X)$ assigns a probability $p_x \geq 0$ to each prize x, with $\sum_{x \in X} p_x = 1$. We write δ_x for "x for sure" and

$$p = \sum_{x \in X} p_x \, \delta_x, \qquad \text{e.g. } p = 0.15\,\delta_a + 0.35\,\delta_b + 0.5\,\delta_c.$$

With money prizes we also write $p = [p_1 : x_1;\ p_2 : x_2; \ldots]$, as in Lecture 1.

Expected value and the St. Petersburg paradox

The old rule: maximize expected value

For a long time, the “rational” rule was to maximize **expected value**:

$$p \succsim q \iff \sum_{x \in X} p_x \, x \;\geq\; \sum_{x \in X} q_x \, x.$$

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St. Petersburg game (opening question)

Toss a coin until the first head. If it comes on toss k, you get $\$2^k$. Probability of that event: 2^{-k}.

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$$\sum_{k=1}^{\infty} 2^{-k} \cdot 2^k = 1 + 1 + 1 + \cdots = \infty$$

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Paradox (Bernoulli, 1954, orig. 1738): an expected-value maximizer would pay *any* price to play, yet you wrote down a small number.

What goes wrong? The pauper's ticket

The magic ticket

A pauper finds a magic lottery ticket: $[50\% : \$1{,}000{,}000;\ 50\% : \$0]$. A rich person offers him **$499,999 for sure**.

- Expected value of the ticket: \$500,000 $>$ \$499,999 $\Rightarrow$ the pauper should **refuse**.

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Is that reasonable?

- Going from \$0 to \$499,999 changes his life (food, clothes, a home).
- Going from \$499,999 to \$1,000,000 changes much less (a third pair of silk pyjamas).
- By keeping the ticket he risks losing a lot (\$0 vs. \$499,999) to gain relatively little.

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The *value* of money is not proportional to the amount: an extra dollar matters less when you are rich.

Bernoulli: maximize expected *utility*

Bernoulli (1738)

Bernoulli (1954, orig. 1738): people maximize the expectation of a **utility of money** $u(x)$, with **decreasing marginal utility** (u concave).

Example: the pauper

$u(\$0) = 0, \quad u(\$499{,}999) = 10, \quad u(\$1{,}000{,}000) = 16.$

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Ticket: $\frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 0 = 8 \; < \; 10 =$ sure offer $\Rightarrow$ the pauper sells.

St. Petersburg with $u(x) = \ln x$

Bernoulli proposed $u(x) = \ln x$. In the St. Petersburg game (no other wealth):

$$\sum_{k \geq 1} 2^{-k} \ln\left(2^k\right) = \ln 2 \sum_{k \geq 1} k \, 2^{-k} = 2 \ln 2 = \ln 4.$$

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With $u = \ln$, the game is worth exactly **\$4** for sure: a finite, small number, like your answers.

Risk aversion and the certainty equivalent

Definitions

For a money lottery p with mean $\mathbb{E}[p] = \sum_x p_x x$ and expected utility $U(p) = \sum_x p_x u(x)$:

- **Certainty equivalent** $CE(p)$: the sure amount with $u(CE(p)) = U(p)$.
- **Risk premium** $RP(p) = \mathbb{E}[p] - CE(p)$.
- The DM is **risk averse** if $\delta_{\mathbb{E}[p]} \succsim p$ for every p, i.e. $RP(p) \geq 0$.

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Example

$u(x) = \sqrt{x}$,
$p = [50\% : \$0;\ 50\% : \$100]$:
$\mathbb{E}[p] = 50,\ U(p) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 10 = 5,$
$CE(p) = 5^2 = 25,$
$RP(p) = 50 - 25 = 25.$

Risk aversion = decreasing marginal utility?

Under expected utility

If people

- maximize expected utility, and
- have decreasing marginal utility (u concave),

then they are **risk averse**: they prefer the expected value of any lottery for sure.

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Under EU, attitudes to risk are entirely carried by the shape of u. Lecture 11 separates them: attitudes to risk can also come from how people **weigh probabilities**.

Expected utility: what can we test?

The problem

- Expected utility is the workhorse model of choice under risk.
- Like ordinary utility, it contains something **unobservable**: the utility $u(x)$ of each prize.
- We must **infer** u from choices between lotteries.

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Definition

A preference $\succsim$ over $\Delta(X)$ has an **expected utility representation** if there is $u : X \to \mathbb{R}$ such that for all $p, q \in \Delta(X)$:

$$p \succsim q \iff U(p) = \sum_{x \in X} p_x \, u(x) \geq \sum_{x \in X} q_x \, u(x) = U(q).$$

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Question: for which preferences does such a u exist? Ordinary utility needed a *weak order*. Is that enough here?

Recap: expected utility

Recap

- **Expected value** fails: the St. Petersburg game and the pauper's ticket.
- **Expected utility** $U(p) = \sum_x p_x u(x)$ resolves both with a concave u (Bernoulli).
- Risk aversion $\iff RP \geq 0 \iff$ u concave (under EU). Example: $u = \sqrt{x}$, $CE = 25$, $RP = 25$.

Open question: which **choice behavior** is equivalent to maximizing expected utility?

3

The axioms of expected utility

A weak order satisfying Independence and Continuity is exactly what expected utility maximization requires, with a utility that is unique up to positive affine transformations.

Which axioms?

Weak order

An expected utility representation is still a **utility representation**, so $\succsim$ must be a **weak order**:

- **Complete:** for all p, q: $p \succsim q$ or $q \succsim p$ (this implies reflexivity, $p \succsim p$).
- **Transitive:** $p \succsim q$ and $q \succsim r$ imply $p \succsim r$.

Which axioms?

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Not enough

Not surprisingly, this is not enough: a weak order can be represented by $U(p) = \sum_x p_x^2 \, u(x)$, which is not EU. We need two more axioms:

1. **Independence**
2. **Continuity**

Both use one operation on lotteries: **mixing**.

Mixtures (compound lotteries)

Definition

For $p, q \in \Delta(X)$ and $\alpha \in [0,1]$, the **mixture** $\alpha p + (1-\alpha)q$ is the lottery with

$$\big(\alpha p + (1-\alpha)q\big)_x = \alpha p_x + (1-\alpha)q_x \qquad \text{for each } x \in X.$$

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Interpretation: a two-stage lottery

- with probability α you play lottery p,
- with probability $1-\alpha$ you play lottery q.

Mixtures (compound lotteries)

Definition

For $p, q \in \Delta(X)$ and $\alpha \in [0,1]$, the **mixture** $\alpha p + (1-\alpha)q$ is the lottery with

$$\big(\alpha p + (1-\alpha)q\big)_x = \alpha p_x + (1-\alpha)q_x \qquad \text{for each } x \in X.$$

Interpretation: a two-stage lottery

- with probability α you play lottery p,
- with probability $1-\alpha$ you play lottery q.

Example

$p = [25\% : a;\ 75\% : b]$, $r = \delta_c$, $\alpha = \frac{1}{2}$:

$$\tfrac{1}{2}p + \tfrac{1}{2}r = [12.5\% : a;\ 37.5\% : b;\ 50\% : c].$$

The Independence axiom: an example

Two lotteries over fruit

- $p = [25\% : \text{apple};\ 75\% : \text{banana}]$,
- $q = [75\% : \text{apple};\ 25\% : \text{banana}]$.
- Suppose you prefer $p \succsim q$.

The Independence axiom: an example

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Two compound lotteries: flip a coin

1. $\frac{1}{2}p + \frac{1}{2}r$: heads, you play p; tails, you get r (say, celery).
2. $\frac{1}{2}q + \frac{1}{2}r$: heads, you play q; tails, you get r.

The Independence axiom: an example

Two lotteries over fruit

- $p = [25\% : \text{apple};\ 75\% : \text{banana}]$,
- $q = [75\% : \text{apple};\ 25\% : \text{banana}]$.
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2. $\frac{1}{2}q + \frac{1}{2}r$: heads, you play q; tails, you get r.

Independence

You should prefer the first: $\frac{1}{2}p + \frac{1}{2}r \succsim \frac{1}{2}q + \frac{1}{2}r$.

- The alternative outcome r is the **same** in both compound lotteries.
- So the choice should depend only on the comparison between p and q.

The Independence axiom

Independence axiom

For all $p, q, r \in \Delta(X)$ and all $\alpha \in (0, 1]$:

$$p \succsim q \iff \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r.$$

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Linearity in probabilities

Independence expresses the **linearity in probabilities** of expected utility. Since $U(\alpha p + (1-\alpha)r) = \alpha U(p) + (1-\alpha)U(r)$:

$$U(p) \geq U(q) \iff \alpha U(p) + (1-\alpha)U(r) \geq \alpha U(q) + (1-\alpha)U(r).$$

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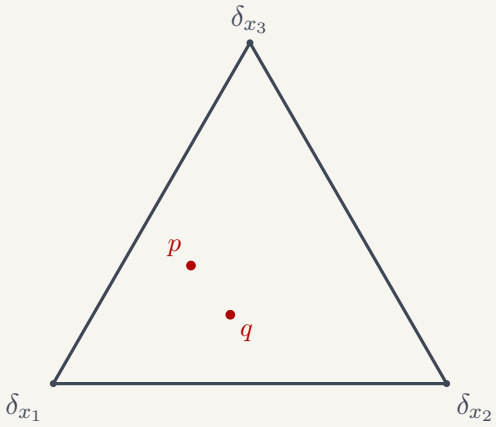
Linearity in probabilities

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$$U(p) \geq U(q) \iff \alpha U(p) + (1-\alpha) U(r) \geq \alpha U(q) + (1-\alpha) U(r).$$

Mixing in a common r does not change your ranking of p and q. The " $\iff$ " also gives the strict version: $p \succ q \Rightarrow \alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) r$.

Independence $\Rightarrow$ indifference curves are parallel straight lines



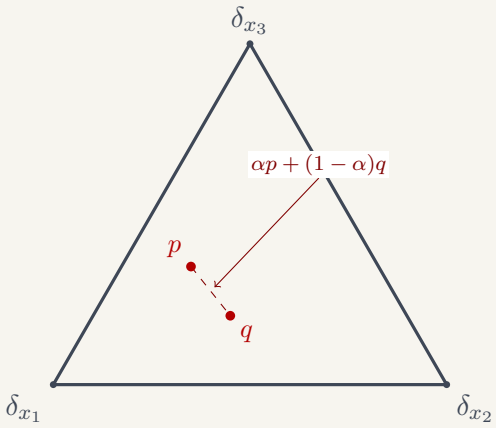
The triangle

Lotteries over three prizes = points of the triangle (simplex $\Delta(X)$); vertices = sure prizes.

Argument

- Let $p \sim q$.

Independence $\Rightarrow$ indifference curves are parallel straight lines



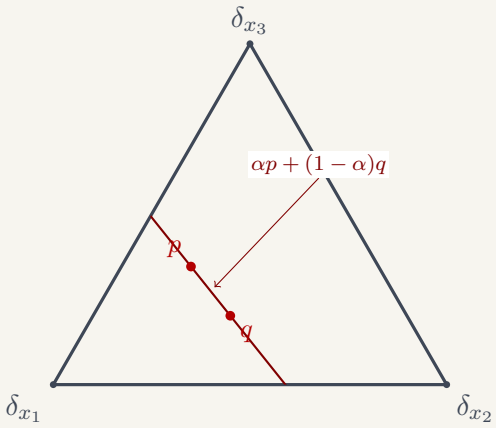
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Lotteries over three prizes = points of the triangle (simplex $\Delta(X)$); vertices = sure prizes.

Argument

- Let $p \sim q$.
- Independence twice: $\alpha p + (1-\alpha)q \ \sim \ \alpha q + (1-\alpha)q = q$. The whole segment is indifferent.

Independence $\Rightarrow$ indifference curves are parallel straight lines



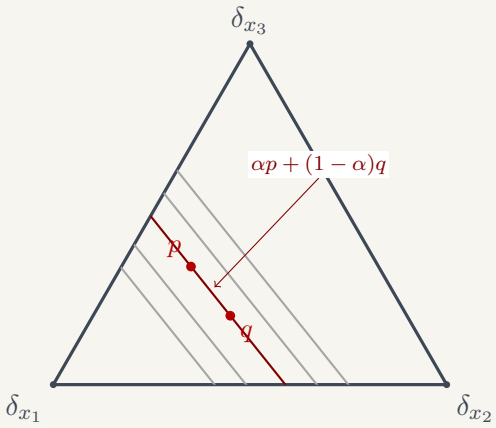
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- So indifference curves are **straight lines**.

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- Let $p \sim q$.
- Independence twice: $\alpha p + (1-\alpha)q \ \sim \ \alpha q + (1-\alpha)q = q$. The whole segment is indifferent.
- So indifference curves are **straight lines**.
- They are also **parallel**: U is linear, so its level sets are parallel lines.

Independence and your Lecture 1 answers

The opening poll of Lecture 1

Kahneman and Tversky (1979, Problems 3–4):

$$
\begin{aligned}
A &= [100\% : \$3000], \\
B &= [80\% : \$4000;\ 20\% : \$0], \\
C &= [25\% : \$3000;\ 75\% : \$0], \\
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Check

$C = \frac{1}{4}A + \frac{3}{4}\delta_0$ and $D = \frac{1}{4}B + \frac{3}{4}\delta_0$.
Independence with $\alpha = \frac{1}{4}$, $r = \delta_0$:

$$A \succsim B \iff C \succsim D.$$

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$C = \frac{1}{4}A + \frac{3}{4}\delta_0$ and $D = \frac{1}{4}B + \frac{3}{4}\delta_0$.
Independence with $\alpha = \frac{1}{4}$, $r = \delta_0$:

$$A \succsim B \iff C \succsim D.$$

Choosing A **and** D (the majority answer) violates **Independence** itself, whatever the utility of money. No choice of u can fix it: Lecture 11 relaxes the axiom.

The Continuity axiom

The second axiom is more technical.

Continuity (Archimedean) axiom

For all $p, q, r \in \Delta(X)$ with $p \succsim q \succsim r$, there exists $\alpha \in [0, 1]$ such that

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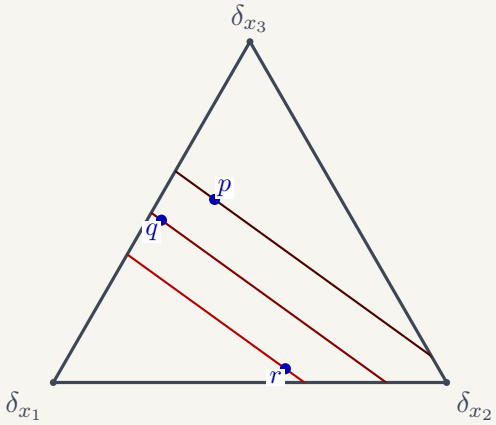
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Interpretation: no prize is infinitely good or infinitely bad

- $p =$ \$10 for sure, $q =$ \$5 for sure, $r =$ death.
- Continuity: for some α close to 1, you are indifferent between q and "\$10, with a tiny risk of death".
- Most people accept this: you cross a street to save \$5.

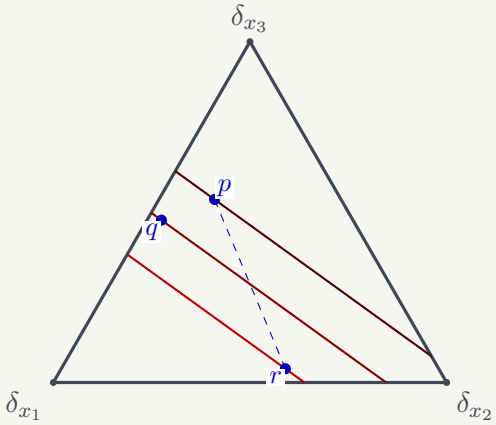
Continuity in the triangle



Reading the picture

- Three indifference lines: $p \succ q \succ r$.

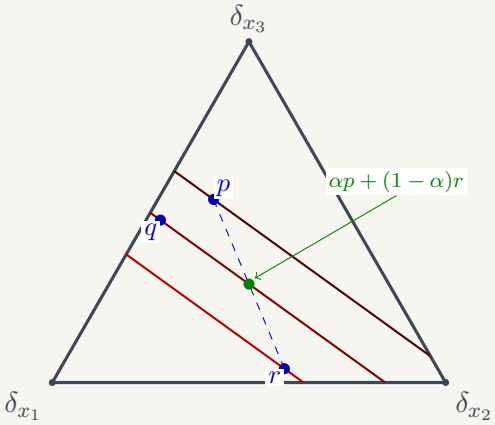
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- Three indifference lines: $p \succ q \succ r$.
- Move along the segment from r to p.

Continuity in the triangle



Reading the picture

- Three indifference lines: $p \succ q \succ r$.
- Move along the segment from r to p.
- Somewhere it **crosses** q's indifference line: there $\alpha p + (1-\alpha)r \sim q$ (here $\alpha = \frac{1}{2}$).

Continuity rules out "jumps" over q's indifference curve.

The expected utility theorem

Theorem (von Neumann–Morgenstern)

Let X be a finite set of prizes and $\succsim$ a binary relation on $\Delta(X)$. Then $\succsim$ is **complete**, **transitive** and satisfies **Independence** and **Continuity** if and only if there exists $u : X \to \mathbb{R}$ such that, for all $p, q \in \Delta(X)$,

$$p \succsim q \iff \sum_{x \in X} p_x \, u(x) \geq \sum_{x \in X} q_x \, u(x).$$

Sources

First proved by von Neumann and Morgenstern (1947); this version follows Herstein and Milnor (1953) and Kreps (1988).

Two directions

- **Necessity** (EU $\Rightarrow$ axioms) is easy: Exercise 2.
- **Sufficiency** (axioms $\Rightarrow$ EU) is harder: three steps.

Proof, step 1: a candidate utility

Standard lotteries

Write $L(\alpha) = \alpha\,\delta_{\bar{x}} + (1-\alpha)\,\delta_{\underline{x}}$ for a **standard lottery**.

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Best and worst prize

- X is finite and $\succsim$ is a weak order, so there is a **best prize** $\bar{x}$ and a **worst prize** $\underline{x}$ (among the sure prizes δ_x).
- By Independence (induction on the number of prizes), $\delta_{\bar{x}} \succsim p \succsim \delta_{\underline{x}}$ for **all** $p \in \Delta(X)$.
- If $\delta_{\bar{x}} \sim \delta_{\underline{x}}$, all lotteries are indifferent and any constant u works. So let $\delta_{\bar{x}} \succ \delta_{\underline{x}}$.

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Continuity defines $U(p)$

By **Continuity** (with $\delta_{\bar{x}} \succsim p \succsim \delta_{\underline{x}}$), for each p there is a number $U(p) \in [0,1]$ with

$$p \;\sim\; L\big(U(p)\big) = U(p)\,\delta_{\bar{x}} + \big(1-U(p)\big)\,\delta_{\underline{x}}.$$

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Claim: $U : \Delta(X) \to [0,1]$ is well defined (step 2), represents $\succsim$ (step 2) and is an expected utility (step 3).

Proof, step 2: U represents $\succsim$

Standard lotteries are ranked by their weight

Let $\alpha > \beta$. Both lotteries are mixtures with a common part $r = \dfrac{\beta\,\delta_{\bar{x}} + (1-\alpha)\,\delta_{\underline{x}}}{1-(\alpha-\beta)}$:

$$L(\alpha) = (\alpha-\beta)\,\delta_{\bar{x}} + \big(1-(\alpha-\beta)\big)\,r, \qquad L(\beta) = (\alpha-\beta)\,\delta_{\underline{x}} + \big(1-(\alpha-\beta)\big)\,r.$$

Since $\delta_{\bar{x}} \succ \delta_{\underline{x}}$, Independence (strict version) gives $L(\alpha) \succ L(\beta)$.

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Consequences

- $U(p)$ is **unique**: two different weights cannot both be indifferent to p.
- For all p, q:

$$p \succsim q \iff L\big(U(p)\big) \succsim L\big(U(q)\big) \iff U(p) \geq U(q).$$

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U represents $\succsim$. It remains to show that U has the expected utility form.

Proof, step 3: U is an expected utility

U is linear in mixtures

Take p, q and $\alpha \in [0,1]$. Using **Independence** twice (replace p, then q, by an indifferent lottery):

$$\begin{aligned} \alpha p + (1-\alpha)q &\sim \alpha\, L\big(U(p)\big) + (1-\alpha)\, q \\ &\sim \alpha\, L\big(U(p)\big) + (1-\alpha)\, L\big(U(q)\big) \;=\; L\big(\alpha U(p) + (1-\alpha)U(q)\big). \end{aligned}$$

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By uniqueness of U (step 2):

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Conclusion

By induction, $U\big(\sum_i \alpha_i p_i\big) = \sum_i \alpha_i U(p_i)$. Since every lottery is $p = \sum_x p_x\, \delta_x$:

$$U(p) = \sum_{x \in X} p_x\, U(\delta_x) = \sum_{x \in X} p_x\, u(x) \qquad \text{with } u(x) := U(\delta_x). \qquad \blacksquare$$

Expected utility numbers are cardinal

Ordinary utility is **ordinal**: any increasing transformation represents the same choices. Not here.

Theorem (uniqueness)

Let $u : X \to \mathbb{R}$ be an expected utility representation of $\succsim$ on $\Delta(X)$. Then $v : X \to \mathbb{R}$ is also one if and only if

$$v(x) = a\,u(x) + b \quad \text{for all } x \in X,$$
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Example

$p = \left[50\% : \$0;\ 50\% : \$100\right]$ vs. δ_{30}.

- $u(x) = \sqrt{x}$: $U(p) = 5 < \sqrt{30} \approx 5.48$, so $\delta_{30} \succ p$. Same with $v = 2\sqrt{x} + 3$: $13 < 13.95$.
- $w = u^2$, i.e. $w(x) = x$ (increasing in u, not affine): $50 > 30$, so $p \succ \delta_{30}$. **Different choices.**

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Under EU, **ratios of utility differences** $\frac{u(x)-u(y)}{u(z)-u(y)}$ are the same in all representations.

Recap: the axioms of expected utility

The vNM theorem

- Weak order $+$ **Independence** $+$ **Continuity** $\iff$ expected utility (vNM).
- Independence $\iff$ linearity in probabilities: **parallel straight** indifference lines in the triangle.

Proof and uniqueness

- Proof: $u(x)$ = weight on the best prize in the standard lottery indifferent to x.
- u is unique up to $a\,u + b$, $a > 0$: utility is **cardinal**.

The Lecture 1 Allais choices (A and D) violate Independence, whatever u.

4

Choice under uncertainty

Savage and Anscombe–Aumann reveal a utility and a subjective probability at the same time, from choices between acts.

Savage's framework

So far

- reveal subjective probabilities, given utility (Ramsey, de Finetti);
- reveal utilities, given objective probabilities (vNM).

Savage (1954) reveals both at once.

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Setup

- A set of **states** Ω (which horse wins), a set of **outcomes** X.
- **Acts:** maps $f : \Omega \to X$ (a bet: "\$10 if Fire Wind wins, \$0 otherwise").
- Preferences $\succsim$ over acts; **no probabilities are given**.

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Theorem (Savage)

$\succsim$ satisfies his seven postulates (P1–P7) iff there exist $u : X \to \mathbb{R}$ and a probability μ on Ω with

$$f \succsim g \iff \int_\Omega u(f(\omega))\, d\mu(\omega) \;\geq\; \int_\Omega u(g(\omega))\, d\mu(\omega).$$

μ is unique; u is unique up to positive affine transformations.

The sure-thing principle

For an event $A \subseteq \Omega$ and acts f, h, write $f_A h$ for the act that equals f on A and h on A^c.

Savage's P2 (sure-thing principle)

For all acts f, g, h, h' and events A:
$f_A h \succsim g_A h \iff f_A h' \succsim g_A h'$.

What happens **outside** A (where f and g agree) does not matter for the comparison: the analogue of Independence.

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If you prefer f to g when you **know** A occurs, and also when you **know** A^c occurs, then you prefer f to g:

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P2 is the axiom that the **Ellsberg paradox** violates (Lecture 12): with unknown probabilities, people's choices do depend on what happens in states where the two acts agree.

The Anscombe–Aumann framework

Setup

Anscombe and Aumann (1963) propose a less general but much simpler setting:

- a **finite** set of states Ω, a finite set of outcomes X;
- acts are maps $f : \Omega \to \Delta(X)$: in each state you receive a **lottery**.

The Anscombe–Aumann framework

Setup

Anscombe and Aumann (1963) propose a less general but much simpler setting:

- a **finite** set of states Ω, a finite set of outcomes X;
- acts are maps $f : \Omega \to \Delta(X)$: in each state you receive a **lottery**.

Horse race + roulette

- "Horse race + roulette": subjective uncertainty about the state, objective risk given the state.
- Clear separation between **objective** and **subjective** probability.
- Acts can be **mixed** state by state: $\big(\alpha f + (1-\alpha)g\big)(\omega) = \alpha f(\omega) + (1-\alpha)g(\omega)$, so the vNM axioms make sense for acts.
- Less general than Savage (it assumes the convex structure of $\Delta(X)$), but much easier to work with: most of the ambiguity literature (Lecture 12) uses it.

The Anscombe–Aumann theorem

Axioms

Identify each lottery $\ell \in \Delta(X)$ with the constant act "ℓ in every state". The axioms:

- **Weak order**, **Independence** and **Continuity**, as in vNM, now for acts and their mixtures.
- **Monotonicity:** if $f(\omega) \succsim g(\omega)$ for every state ω, then $f \succsim g$.
- **Non-degeneracy:** $f \succ g$ for some acts f, g.

Theorem (Anscombe–Aumann)

$\succsim$ satisfies these axioms iff there exist $u : X \to \mathbb{R}$ (non-constant) and a probability μ on Ω such that

$$f \succsim g \iff \sum_{\omega \in \Omega} \mu(\omega)\, U\big(f(\omega)\big) \;\geq\; \sum_{\omega \in \Omega} \mu(\omega)\, U\big(g(\omega)\big), \qquad U(\ell) = \sum_{x \in X} \ell_x\, u(x).$$

μ is unique; u is unique up to positive affine transformations.

Recap: Assumption 2, made precise

Three representation theorems

Setting	Objects of choice	Key axiom	Representation
Risk (vNM)	lotteries $p \in \Delta(X)$	Independence	$\sum_x p_x u(x)$
Uncertainty (Savage)	acts $f : \Omega \to X$	Sure-thing (P2)	$\int u(f)\, d\mu$
Uncertainty (AA)	acts $f : \Omega \to \Delta(X)$	Independence	$\sum_\omega \mu(\omega) U(f(\omega))$

"Robinson knows p and maximizes expected utility" is not a vague slogan: it is equivalent to a short list of **testable axioms on choices**. The next lectures test them one by one.

Exercises

1. Easy

Let $u(x) = \sqrt{x}$ and $p = [25\% : \$0;\ 75\% : \$400]$. Compute $\mathbb{E}[p]$, $U(p)$, $CE(p)$ and $RP(p)$. Is this DM risk averse?

2. Intermediate

Show that if $\succsim$ has an expected utility representation, then it satisfies Independence and Continuity (necessity in the vNM theorem).

3. Advanced

Prove the uniqueness theorem: u and v represent the same $\succsim$ iff $v = au + b$ with $a > 0$. *Hint for "$\Rightarrow$": use the standard lotteries $L(\alpha)$ from the proof.*

4. Discussion

Risk or uncertainty? Which view of probability (frequentist, logical, subjective) fits best? (a) a roulette spin; (b) your car insurance premium; (c) "a war in Europe in the next ten years"; (d) the success of a start-up.

Final take-aways

1. Decision theory takes a **subjective, pragmatist** view of probability: beliefs are revealed by choices.
2. **Expected utility** resolves the St. Petersburg paradox; risk aversion = concave u, measured by CE and RP.
3. vNM: weak order $+$ **Independence** $+$ **Continuity** $\iff$ expected utility, with a **cardinal** u.
4. Savage and Anscombe–Aumann extend this to **subjective** probabilities.
5. Each axiom is **testable**. Independence already fails in your own Lecture 1 answers: Lecture 11.

References I

Anscombe, F. J. and Aumann, R. J. (1963). A definition of subjective probability. *Annals of Mathematical Statistics*, 34(1):199–205.

Bernoulli, D. (1954). Exposition of a new theory on the measurement of risk. *Econometrica*, 22(1):23–36. Translation of the Latin original of 1738.

de Finetti, B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincaré*, 7(1):1–68.

Halevy, Y. (2007). Ellsberg revisited: An experimental study. *Econometrica*, 75(2):503–536.

Herstein, I. N. and Milnor, J. (1953). An axiomatic approach to measurable utility. *Econometrica*, 21(2):291–297.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–291.

Keynes, J. M. (1921). *A Treatise on Probability*. Macmillan, London.

Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money*. Macmillan, London.

Keynes, J. M. (1937). The general theory of employment. *Quarterly Journal of Economics*, 51(2):209–223.

Knight, F. H. (1921). *Risk, Uncertainty and Profit*. Houghton Mifflin, Boston.

Kreps, D. (1988). *Notes On The Theory Of Choice*. Westview Press.

Rabin, M. (2000). Risk aversion and expected-utility theory: A calibration theorem. *Econometrica*, 68(5):1281–1292.

Ramsey, F. P. (1931). Truth and probability. In Braithwaite, R. B., editor, *The Foundations of Mathematics and Other Logical Essays*. Kegan Paul, Trench, Trubner, London.

Savage, L. J. (1954). *The Foundations of Statistics*. Wiley, New York.

von Neumann, J. and Morgenstern, O. (1947). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, 2nd edition.

Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus*. Kegan Paul, Trench, Trubner, London.