BEHAVIORAL ECONOMICS · LECTURE 12

Ambiguity: when probabilities are unknown

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Master Economics – Winter 2026/27

Before we start: two urns

Urn K (known)

100 balls,
50 red and 50 black.

Urn U (unknown)

100 balls, red and black
in an **unknown** proportion.

Your answers

One ball will be drawn from the urn you choose. Write down your answers.

1. You win €100 if the ball is **red**. Which urn do you draw from? (K / U / indifferent)
2. You win €100 if the ball is **black**. Which urn do you draw from? (K / U / indifferent)

Learning objectives

By the end of this lecture you should be able to:

- Distinguish **risk** (known probabilities) from **ambiguity** (unknown probabilities).
- State Savage's **sure-thing principle** and show that the Ellsberg choices violate it.
- Compute **maxmin expected utility** and explain why it values **hedging**.
- State the axioms of Gilboa and Schmeidler (1989) and their representation theorem.
- Compare α-maxmin, smooth and variational models on the Ellsberg urn.

Roadmap

Subjective expected utility

The Ellsberg paradox

Maxmin expected utility

Beyond maxmin

Where we are in the course

The assumptions of the Robinson model

0. **Consistent preferences:** Robinson's choices can be described by a utility function u
1. **Costless optimization:** he always finds the option that maximizes u
2. **Correct beliefs & expected utility:** he knows p and maximizes expected utility
3. **Absolute, self-regarding consumption:** only his own (b, y) enters u
4. **Stable tastes:** the same u in both periods

Lectures 10–11 kept the probabilities p **known** and relaxed the *evaluation* (expected utility). Today we drop the assumption that Robinson **knows** p at all: he faces *ambiguity*.

1

Subjective expected utility

Even without given probabilities, Savage's axioms let us derive a subjective prior from choices; the sure-thing principle is the key axiom.

The Savage framework

Primitives

Following Savage (1954):

- a set of **states** Ω, one of which is true (finite in our examples); **events** are subsets $A \subseteq \Omega$;
- a set of **outcomes** X (e.g. amounts of money);
- **acts** $f : \Omega \to X$: $f(\omega)$ is what you get if state ω is true;
- a preference relation $\succsim$ over acts.

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Theorem (Savage, 1954)

$\succsim$ satisfies Savage's axioms P1–P7 iff there are a utility $u : X \to \mathbb{R}$ and a probability p on Ω such that

$$f \succsim g \iff \int_\Omega u(f(\omega))\, dp(\omega) \geq \int_\Omega u(g(\omega))\, dp(\omega).$$

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$$f \succsim g \iff \int_\Omega u(f(\omega))\, dp(\omega) \geq \int_\Omega u(g(\omega))\, dp(\omega).$$

No probability is given: p is **subjective**, derived from choices. Even so, the decision maker behaves *as if* she had one prior p and maximized expected utility (SEU).

The sure-thing principle (Savage's P2)

Notation

For acts f, g and an event A, let $f_A g$ be the act that equals f on A and g on A^c:

$$f_A g(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in A, \\ g(\omega) & \text{if } \omega \in A^c. \end{cases}$$

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Sure-thing principle

For all acts f, g, h, k and every event A:

$$f_A h \succsim g_A h \iff f_A k \succsim g_A k.$$

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Interpretation

If f and g agree on A^c, the comparison should depend only on what happens on A: what they have *in common* on A^c (whether h or k) is irrelevant.

The sure-thing principle plays for acts the role that the **independence axiom** plays for lotteries (Lecture 10).

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The Ellsberg paradox

People prefer known odds both for a bet and for its complement: no single prior can explain this.

Back to the two urns

Under SEU

Let $p_U(R)$ and $p_U(B) = 1 - p_U(R)$ be your subjective probabilities for urn U. Under SEU with $u(100) > u(0)$:

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- Choosing **K for red** means $\frac{1}{2} > p_U(R)$.
- Choosing **K for black** means $\frac{1}{2} > p_U(B)$.

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Our class

____ chose K–K, ____ chose U–U, ____ were indifferent or mixed.

Preferring the known urn for **both** colours cannot be explained by any single prior: people dislike not knowing the probabilities. This is **ambiguity aversion**.

The three-colour urn (Ellsberg, 1961)

The urn and four bets

An urn contains **90 balls**: 30 red, and 60 black or yellow in an **unknown** proportion. One ball is drawn: $\Omega = \{r, b, y\}$.

	Red (r)	Black (b)	Yellow (y)
	30 balls	60 balls, unknown mix	
f^1: bet on red	€100	€0	€0
f^2: bet on black	€0	€100	€0
f^3: bet on red or yellow	€100	€0	€100
f^4: bet on black or yellow	€0	€100	€100

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f^1 or f^2? And f^3 or f^4?

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Your choice

f^1 or f^2? And f^3 or f^4?

Typical choices: $f^1 \succ f^2$ and $f^4 \succ f^3$.

Why this violates the sure-thing principle

Rewriting the four bets

Take $A = \{r, b\}$ and the act g that pays €100 whatever the colour. Then (look at the table):

$$f^1 = f^1_A\, 0, \qquad f^2 = f^2_A\, 0, \qquad f^3 = f^1_A\, g, \qquad f^4 = f^2_A\, g,$$

where 0 is the act paying €0. The pairs differ **only** on the column “yellow”, and within each pair the two acts agree there.

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Sure-thing principle vs. observed choices

- Sure-thing principle:
  $f^1_A 0 \succ f^2_A 0 \iff f^1_A g \succ f^2_A g$,
  i.e. $f^1 \succ f^2 \iff f^3 \succ f^4$.
- Observed: $f^1 \succ f^2$ but $f^4 \succ f^3$. Contradiction.

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Same conclusion with probabilities

$f^1 \succ f^2 \Rightarrow p(r) > p(b)$;
$f^4 \succ f^3 \Rightarrow p(b) + p(y) > p(r) + p(y) \Rightarrow p(b) > p(r)$.

Interpreting the Ellsberg paradox

Known odds are preferred

- **Red** has a *known* probability ($\frac{1}{3}$); black and yellow do not. Likewise “black or yellow” ($\frac{2}{3}$) is known, “red or yellow” is not.
- People prefer bets with known odds: **ambiguity aversion**.
- So preferences depend not only on outcomes and *beliefs*, but on **how confident** one is in these beliefs.

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Evidence

- The Ellsberg pattern is robust in the lab; Halevy (2007) finds ambiguity aversion closely linked to failures to reduce **compound lotteries**.
- Not universal: ambiguity *seeking* is common for **unlikely gains**.

No single prior can explain the Ellsberg choices: SEU must be extended.

Recap: risk, ambiguity and Ellsberg

Risk, ambiguity, SEU

- **Risk:** probabilities known (Lectures 10–11). **Ambiguity:** probabilities unknown.
- **SEU** (Savage): one subjective prior p, expected utility under p.
- SEU implies the **sure-thing principle**.

Ellsberg

- **Ellsberg:** bets with known odds are preferred, for both colours/complements. No single p fits, so the sure-thing principle fails.

Next: a model with many *priors.*

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Maxmin expected utility

Evaluating each act with its own worst-case prior explains Ellsberg and reveals a taste for hedging.

Many priors: maxmin expected utility

Idea

The decision maker does not have one prior but a **set of priors** Π, and is cautious.

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Ellsberg urn

If the urn has k black balls ($k = 0, \dots, 60$), the prior is

$$p_k(r) = \tfrac{30}{90} = \tfrac{1}{3}, \quad p_k(b) = \tfrac{k}{90}, \quad p_k(y) = \tfrac{60-k}{90},$$

and $\Pi = \{p_0, \dots, p_{60}\}$ (p_0: only yellow, p_{60}: only black).

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Maxmin expected utility (MEU)

Each act is evaluated with **its own worst-case prior**,

$$\begin{aligned} V(f) &= \min_{p \in \Pi} \mathbb{E}_p\big[u(f)\big] \\ &= \min_{p \in \Pi} \sum_{\omega} p(\omega)\, u\big(f(\omega)\big). \end{aligned}$$

- If $\Pi = \{p\}$ is a singleton, MEU = SEU.
- A formal version of the *precautionary principle* (Hans Jonas): plan for the worst plausible case.

Worked example: Ellsberg under maxmin

Maxmin values of the four bets

Take $u(x) = x$. Under p_k, $p_k(b) = k/90$ ranges over $[0, \frac{2}{3}]$ and $p_k(y) = \frac{2}{3} - p_k(b)$.

Act	$\mathbb{E}_{p_k}[u(f)]$	worst prior	$V(f)$
f^1: red	$100 \cdot \frac{1}{3}$	any	33.3
f^2: black	$100 \cdot \frac{k}{90}$	$k = 0$ (no black)	0
f^3: red or yellow	$100 \cdot \left(\frac{1}{3} + \frac{60-k}{90}\right)$	$k = 60$ (no yellow)	33.3
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So $V(f^1) > V(f^2)$ and $V(f^4) > V(f^3)$: MEU explains the Ellsberg choices.

The worst-case prior for f^2 ($k = 0$) differs from the one for f^3 ($k = 60$). No single prior does this: that is exactly how MEU escapes the sure-thing principle.

A richer framework: Anscombe–Aumann

Anscombe–Aumann acts

To state the axioms, Gilboa and Schmeidler (1989) use the framework of Anscombe and Aumann (1963):

- acts map states into **lotteries**: $f : \Omega \to \Delta(X)$ (e.g. “if black, toss a fair coin for €100”);
- a **constant act** c gives the same lottery in every state (pure risk);
- **mixtures** are taken state by state: for $\alpha \in [0, 1]$,

$$(\alpha f + (1 - \alpha)g)(\omega) = \alpha f(\omega) + (1 - \alpha)g(\omega).$$

- u is a vNM utility over lotteries (Lecture 10): $u(\ell) = \sum_x \ell(x)\, u(x)$.

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Why? Mixing lets us ask whether the decision maker likes to **hedge** against ambiguity.

Worked example: the value of hedging

Two ambiguous bets and their mixture

Let f_b = €100 if black, f_y = €100 if yellow (zero otherwise), and $h = \frac{1}{2}f_b + \frac{1}{2}f_y$.

	r	b	y	MEU value V
f_b	€0	€100	€0	$\min_k 100\frac{k}{90} = 0$
f_y	€0	€0	€100	$\min_k 100\frac{60-k}{90} = 0$
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Hedging and independence

- $f_b \sim f_y$, yet the mixture is **strictly better**: $h \succ f_b$.
- **Independence** (Lecture 10) would require $h \sim f_b$: MEU violates independence.

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Mixing two ambiguous bets cancels the ambiguity: the probability of winning is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ under **every** prior. An ambiguity-averse decision maker values this hedge.

The Gilboa–Schmeidler axioms (1/2)

Preferences $\succsim$ over Anscombe–Aumann acts f, g, h:

Weak order

$\succsim$ is complete and transitive.

Continuity

If $f \succ g \succ h$, there are $\alpha, \beta \in (0, 1)$ with $\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h$.

Monotonicity

If $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$ (comparing the lotteries as constant acts), then $f \succsim g$.

Non-degeneracy

$f \succ g$ for some acts f, g.

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Non-degeneracy

$f \succ g$ for some acts f, g.

But not independence! It is replaced by two weaker axioms.

The Gilboa–Schmeidler axioms (2/2)

Certainty independence

For all acts f, g, every **constant** act c and $\alpha \in (0,1)$:

$$f \succsim g \iff \alpha f + (1-\alpha)c \succsim \alpha g + (1-\alpha)c.$$

Mixing with a constant act adds pure risk: it cannot hedge any ambiguity, so the ranking should not change.

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Uncertainty aversion

For all acts f, g and $\alpha \in (0,1)$:

$$f \sim g \implies \alpha f + (1-\alpha)g \succsim f.$$

A preference for **hedging**: exactly what we saw with $h = \frac{1}{2} f_b + \frac{1}{2} f_y$.

Representation theorem

Theorem (Gilboa and Schmeidler, 1989)

$\succsim$ satisfies weak order, continuity, monotonicity, non-degeneracy, certainty independence and uncertainty aversion **if and only if** there are a utility $u : X \to \mathbb{R}$ (non-constant, extended to lotteries by expectation) and a nonempty, closed, convex set Π of probabilities on Ω such that

$$f \succsim g \iff \min_{p \in \Pi} \mathbb{E}_p\big[u(f)\big] \geq \min_{p \in \Pi} \mathbb{E}_p\big[u(g)\big].$$

Moreover, u is unique up to a positive affine transformation and Π is unique.

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Moreover, u is unique up to a positive affine transformation and Π is unique.

Both the utility u **and** the set of priors Π are revealed by choices, as in Savage. The set Π measures *how much* ambiguity the decision maker perceives.

Application: no trade under ambiguity

Setup

An asset pays €1 if the ball is **black** (Ellsberg urn) and costs q. A risk-neutral investor ($u(x) = x$) can buy one unit, sell one unit, or do nothing (value 0). Beliefs: $p(b) \in [\underline{\pi}, \overline{\pi}]$.

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Maxmin values

- **Buy:** $V = \min_p p(b) - q = \underline{\pi} - q$. Worth it iff $q < \underline{\pi}$.
- **Sell:** $V = \min_p (q - p(b)) = q - \overline{\pi}$. Worth it iff $q > \overline{\pi}$.

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Ellsberg set vs. SEU

- With the full Ellsberg set, $[\underline{\pi}, \overline{\pi}] = [0, \frac{2}{3}]$: **no trade** at any price $q \in [0, \frac{2}{3}]$.
- An SEU investor with prior $p(b)$ trades at **every** price except $q = p(b)$.

Application: no trade under ambiguity

Setup

An asset pays €1 if the ball is **black** (Ellsberg urn) and costs q. A risk-neutral investor ($u(x) = x$) can buy one unit, sell one unit, or do nothing (value 0). Beliefs: $p(b) \in [\underline{\pi}, \overline{\pi}]$.

Maxmin values

- **Buy:** $V = \min_p p(b) - q = \underline{\pi} - q$. Worth it iff $q < \underline{\pi}$.
- **Sell:** $V = \min_p (q - p(b)) = q - \overline{\pi}$. Worth it iff $q > \overline{\pi}$.

Ellsberg set vs. SEU

- With the full Ellsberg set, $[\underline{\pi}, \overline{\pi}] = [0, \frac{2}{3}]$: **no trade** at any price $q \in [0, \frac{2}{3}]$.
- An SEU investor with prior $p(b)$ trades at **every** price except $q = p(b)$.

Ambiguity creates an interval of prices with no trade (Dow and Werlang, 1992): a candidate explanation for limited stock-market participation and home bias.

Recap: maxmin expected utility

The model

- MEU: $V(f) = \min_{p \in \Pi} \mathbb{E}_p[u(f)]$; each act is judged by its **own** worst prior.
- It explains Ellsberg, and reduces to SEU if $\Pi = \{p\}$.

Axioms and consequences

- Axioms: independence is weakened to **certainty independence**, plus **uncertainty aversion** (= taste for hedging).
- Consequence: no-trade intervals in asset markets.

But: is the worst case all that matters?

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Beyond maxmin

The alpha-maxmin, smooth and variational models relax the extreme pessimism of maxmin while separating beliefs from attitudes.

Limits of maxmin

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- MEU is **extremely pessimistic**: with the full Ellsberg set, a bet on black is worth €0.
- It only looks at the **worst** prior: two acts with the same worst case are equivalent, however good the other scenarios.
- It cannot produce ambiguity **seeking** (e.g. lottery-like bets on start-ups, unlikely gains).

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α-maxmin

Weigh the worst and the best case (Ghirardato et al., 2004):

$$V_\alpha(f) = \alpha \min_{p \in \Pi} \mathbb{E}_p[u(f)] + (1-\alpha) \max_{p \in \Pi} \mathbb{E}_p[u(f)],$$

with $\alpha \in [0, 1]$.

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Ellsberg

$V_\alpha(f^1) = 33.3,\ V_\alpha(f^2) = \alpha \cdot 0 + (1-\alpha) \cdot 66.7.$
So $f^1 \succ f^2 \iff \alpha > \frac{1}{2}$. $\alpha = 1$ is MEU; $\alpha < \frac{1}{2}$ is ambiguity seeking.

Smooth ambiguity (Klibanoff et al., 2005)

Model

Put a **second-order prior** μ on the set of priors and an **ambiguity attitude** ϕ (increasing):

$$V(f) = \sum_{p \in \Pi} \mu(p)\ \phi\Big(\mathbb{E}_p[u(f)]\Big) \qquad \text{(an integral if } \Pi \text{ is infinite).}$$

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Beliefs vs. attitude

- μ describes ambiguity (beliefs about which p is true); ϕ describes the **attitude** towards it:
  - ϕ **concave**: ambiguity averse; ϕ **convex**: ambiguity seeking;
  - ϕ linear: SEU with the average prior $\bar{p} = \sum_p \mu(p)\, p$.
- Like risk aversion (u concave), but over the **expected utilities** of the different scenarios.

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Pros

Smooth, so convenient in applications (portfolio choice, macro).

Cons

Why stop at a second-order prior (*infinite regress*)? The axiomatization uses bets on which p is true, which are hard to observe.

Worked example: smooth ambiguity in Ellsberg

Certainty equivalents of the four bets

Ellsberg urn, $u(x) = x$. Second-order prior: $\mu(p_0) = \mu(p_{60}) = \frac{1}{2}$ (all yellow or all black). $\phi(t) = \sqrt{t}$.

Act	$\mathbb{E}_{p_0}[u]$ (no black)	$\mathbb{E}_{p_{60}}[u]$ (all black)	certainty equivalent $\phi^{-1}(V)$
f^1: red	33.3	33.3	33.3
f^2: black	0	66.7	$\left(\frac{1}{2}\sqrt{66.7}\right)^2 = 16.7$
f^3: red or yellow	100	33.3	$\left(\frac{1}{2}\sqrt{100} + \frac{1}{2}\sqrt{33.3}\right)^2 = 62.2$
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Both bets with unknown odds lose value: $f^1 \succ f^2$ and $f^4 \succ f^3$.

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Under $\bar{p}$ (here $\bar{p}(b) = \frac{1}{3}$) all four certainty equivalents would be 33.3 or 66.7 in pairs: the **concavity of** ϕ alone creates the Ellsberg pattern.

Variational preferences (Maccheroni et al., 2006)

Model

Consider **all** priors, but penalize implausible ones with a cost $c(p) \geq 0$ (convex, $\min_p c(p) = 0$):

$$V(f) = \min_{p \in \Delta(\Omega)} \Big\{ \mathbb{E}_p\big[u(f)\big] + c(p) \Big\}.$$

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Special cases

- **Maxmin** is the special case $c(p) = 0$ if $p \in \Pi$, $c(p) = +\infty$ otherwise.
- **Multiplier preferences** (Hansen and Sargent, 2001): $c(p) = \theta\, R(p \,\|\, q)$, the relative entropy from a reference prior q; $\theta \to \infty$ gives SEU with q.
- All variational preferences are ambiguity averse; a **smaller** cost c means **more** ambiguity aversion.

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Example

Priors with $p(r) = \frac{1}{3}$, $x \equiv p(b) \in [0, \frac{2}{3}]$, cost $c = 300\,(x - \frac{1}{3})^2$:

- f^2: $\min_x \{100x + 300(x - \frac{1}{3})^2\}$ at $x = \frac{1}{6}$:
  $V = 16.7 + 8.3 = 25 < 33.3 = V(f^1)$;
- f^3: $\min_x \{100(1 - x) + 300(x - \frac{1}{3})^2\}$ at $x = \frac{1}{2}$:
  $V = 50 + 8.3 = 58.3 < 66.7 = V(f^4)$.

Recap: models of ambiguity

Five models on the Ellsberg urn

Model	Value of act f	Ambiguity attitude	Ellsberg?
SEU (Savage)	$\mathbb{E}_p[u(f)]$	neutral	no
Maxmin (Gilboa–Schmeidler)	$\min_{p\in\Pi} \mathbb{E}_p[u(f)]$	extreme aversion	yes
α-maxmin	$\alpha \min_\Pi + (1-\alpha)\max_\Pi$	α	iff $\alpha > \frac{1}{2}$
Smooth (KMM)	$\sum_p \mu(p)\,\phi(\mathbb{E}_p[u(f)])$	curvature of ϕ	iff ϕ concave*
Variational (MMR)	$\min_p\{\mathbb{E}_p[u(f)] + c(p)\}$	size of c	yes, if c not flat

* For the Ellsberg urn with a non-degenerate μ.

All models separate **beliefs** (which priors are plausible) from **attitude** (how much the decision maker cares about not knowing). They differ in how they aggregate across priors.

Exercises

1. (Easy)

Maxmin with a smaller set: $p(r) = \frac{1}{3}$ and $p(b) \in [\frac{1}{6}, \frac{1}{2}]$. With $u(x) = x$, compute $V(f^1), \ldots, V(f^4)$. Does the decision maker show the Ellsberg pattern?

2. (Intermediate)

α-maxmin with the full Ellsberg set. Find all α such that $f^1 \succ f^2$ **and** $f^4 \succ f^3$.

3. (Intermediate)

Smooth model of the worked example, with (a) $\phi(t) = t$ and (b) $\phi(t) = t^2$. Compute the certainty equivalents of f^1 and f^2. Interpret.

4. (Advanced)

Show that maxmin preferences satisfy uncertainty aversion: if $V(f) = V(g)$, then $V(\alpha f + (1-\alpha)g) \geq V(f)$. *Hint:* $\mathbb{E}_p[u(\cdot)]$ *is linear in the act, and a minimum of a sum is at least the sum of the minima.*

Final take-aways

1. Under **ambiguity**, probabilities are unknown; SEU still assumes a single (subjective) prior.
2. The **Ellsberg paradox**: people prefer known odds for a bet *and* for its complement. No single prior fits; the **sure-thing principle** fails.
3. **Maxmin EU** evaluates each act with its worst-case prior; it is characterized by **certainty independence** and a taste for **hedging**.
4. α-maxmin, smooth and variational models relax the extreme pessimism while separating **beliefs** from **attitudes**.
5. Ambiguity matters for markets: no-trade intervals, precaution in policy.

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