BEHAVIORAL ECONOMICS · LECTURE 18

# Time preferences and dynamic inconsistency

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Master Economics – Winter 2026/27

# Before we start: two quick choices

Write down your answers.

**Question 1**

Which do you prefer?

€100 **today** or €110 **in one week**

**Question 2**

Which do you prefer?

€100 **in 52 weeks** or €110 **in 53 weeks**

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*Same trade-off (one more week for €10 more), only the date differs. Did you choose the same way?*

## Learning objectives

By the end of this lecture you should be able to:

- Explain **dynamic (in)consistency** and Strotz's result that only exponential discounting is consistent.
- State the axioms behind **exponential discounting**, in particular **stationarity**.
- Show why exponential discounting rules out **preference reversals**, and how the $\beta$–$\delta$ **model** produces them.
- Solve a 3-period consumption–savings problem under **commitment**, for a **naive** and for a **sophisticated** agent.
- Explain why a time-inconsistent agent who anticipates it **demands commitment**.

# Roadmap

Dynamic inconsistency

Exponential discounting

Quasi-hyperbolic discounting

Consumption and savings with present bias

# Where we are in the course

**The assumptions of the Robinson model**

0. **Consistent preferences:** Robinson's choices can be described by a utility function $u$
1. **Costless optimization:** he always finds the option that maximizes $u$
2. **Correct beliefs & expected utility:** he knows $p$ and maximizes expected utility
3. **Absolute, self-regarding consumption:** only his own $(b, y)$ enters $u$
4. **Stable tastes:** the same $u$ in both periods

The second approach to Assumption 4: Robinson's tastes $u$ are fixed, but **how he weighs the future depends on where he stands**. His period-1 plan for period 2 is not what period-2 Robinson wants.

# 1

# Dynamic inconsistency

Unless discounting is exponential, the plan chosen today is not the plan carried out tomorrow.

# Discounting and the consistency of plans

**Setup**

At date $t$, the decision maker evaluates a consumption stream $\mathbf{x} = (x_1, x_2, \ldots, x_T)$ by

$$U_t(\mathbf{x}) \;=\; \sum_{\tau=t}^{T} D(\tau - t)\, u(x_\tau), \qquad D(0) = 1,$$

where $D(\cdot) > 0$ is a **discount function** of the delay $\tau - t$.

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**Question**

For which $D$ is a plan chosen at $t = 1$ still optimal when date $t > 1$ arrives?

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**Question**

For which $D$ is a plan chosen at $t = 1$ still optimal when date $t > 1$ arrives?

**Strotz (1955)**

Plans are always carried out (time consistency) if and only if $D(\tau) = \delta^\tau$ for some $\delta > 0$.

Otherwise the decision maker will rationally **change her plan**, and she may know it in advance.

# Worked example: a plan that will not be carried out

**The numbers**

Linear utility; $D(0) = 1$, $D(1) = 0.5$, $D(2) = 0.45$ (not exponential: $0.45 \neq 0.5^2$).

Choice: **€10 at date 2** or **€15 at date 3**.

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**At date 1**

$D(1) \cdot 10 = 5 \; < \; D(2) \cdot 15 = 6.75$. $\Rightarrow$ Plan: *wait for the 15.*

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**At date 2**

$D(0) \cdot 10 = 10 \; > \; D(1) \cdot 15 = 7.5. \Rightarrow$ *Take the 10.*

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**With exponential discounting**

With $D(\tau) = \delta^\tau$ both comparisons read $\delta^k \cdot 10$ vs. $\delta^{k+1} \cdot 15$, i.e. $10$ vs. $15\,\delta$, at every date $k$.

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**At date 2**

$D(0) \cdot 10 = 10 \; > \; D(1) \cdot 15 = 7.5$. $\Rightarrow$ *Take the 10.*

A non-exponential discount function makes the **same** trade-off look different from different dates.

# Strotz's two strategies

Faced with this, the decision maker can follow one of two strategies (Strotz, 1955):

**1. Precommitment**

Bind the future self, *"either irrevocably or by contriving a penalty for his future self if he should misbehave."*

**2. Consistent planning**

*"resign himself to the fact of intertemporal conflict and decide that his 'optimal' plan at any date is a will-o'-the-wisp which cannot be attained, and learn to select the present action which will be best in the light of future disobedience."*

# Precommitment: Ulysses and the Sirens

**Commitment devices**

If future selves will deviate, the current self may want **commitment devices**:

- illiquid assets, pensions, forced savings;
- contracts, deadlines, rules;
- restricting choice sets to avoid temptation (Lecture 17: menus).

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**Strotz's image**

**Ulysses** has himself tied to the mast, so that he can hear the Sirens without steering towards them.

Bind yourself **now** to avoid a **predictable** future temptation; Robinson pre-pays the yoga class.

# Consistent planning: an intrapersonal game

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**Formally**

An **intrapersonal dynamic game** (Peleg and Yaari, 1973):

- each date-$t$ self is a **player** with its own objective $U_t$;
- each self chooses today's action, anticipating the strategies of later selves;
- the solution concept is **subgame-perfect equilibrium**, which replaces "the optimal plan".

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A self who does **not** anticipate future deviations is called **naive** (Strotz: myopic); one who does is **sophisticated**.

# 2

# Exponential discounting

Exponential discounting is characterized by stationarity, and systematic preference reversals violate it.

# Discounted utility: axioms

**Axioms**

Preferences $\succsim$ over consumption streams $\mathbf{x} = (x_1, x_2, \dots)$, viewed from date 1.

1. **Weak order and continuity:** $\succsim$ is complete, transitive and continuous.
2. **Separability:** how two streams are ranked does not depend on the periods in which they **coincide**.
3. **Monotonicity:** if $x_t \geq y_t$ for all $t$, then $\mathbf{x} \succsim \mathbf{y}$.

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**Theorem (Debreu, 1960; at least three periods)**

$\succsim$ satisfies 1–3 iff it is represented by $U(\mathbf{x}) = \sum_t u_t(x_t)$ with continuous, increasing $u_t$.

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**Stable tastes**

Writing $u_t = D(t-1)\,u$ adds that the **same** $u$ applies in every period: *stable tastes*, Assumption 4.

**Theorem (Debreu, 1960; at least three periods)**

$\succsim$ satisfies 1–3 iff it is represented by $U(\mathbf{x}) = \sum_t u_t(x_t)$ with continuous, increasing $u_t$.

Axioms 1–3 say nothing about **how** the future is discounted. That is the job of the next axiom.

# Exponential discounting: stationarity

**Axioms 4 and 5**

4. **Stationarity:** for all streams and all $c$,

$$(c, x_1, x_2, \dots) \succsim (c, y_1, y_2, \dots) \iff (x_1, x_2, \dots) \succsim (y_1, y_2, \dots).$$

   *Delaying two streams by one period does not change their ranking.*

5. **Impatience:** if $b > a$, then $(b, a, z_3, z_4, \dots) \succ (a, b, z_3, z_4, \dots)$. *The better outcome is preferred sooner.*

# Exponential discounting: stationarity

**Axioms 4 and 5**

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   *Delaying two streams by one period does not change their ranking.*
5. **Impatience:** if $b > a$, then $(b, a, z_3, z_4, \dots) \succ (a, b, z_3, z_4, \dots)$. *The better outcome is preferred sooner.*

**Theorem (Koopmans 1960; textbook form)**

$\succsim$ satisfies 1–5 iff there are a continuous increasing $u$ and $\delta \in (0, 1)$ with

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{t \geq 1} \delta^{t-1} u(x_t) \geq \sum_{t \geq 1} \delta^{t-1} u(y_t).$$

# Violating stationarity

**A choice involving immediate consumption**

1. salad or burger for lunch **today**;
2. 2 squirts of juice **now** or 3 squirts **in 10 minutes**.

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**The same choice, delayed**

1. salad or burger for lunch **next Tuesday**;
2. 2 squirts of juice **in 10 minutes** or 3 squirts **in 20 minutes**.

# Violating stationarity

**A choice involving immediate consumption**

1. salad or burger for lunch **today**;
2. 2 squirts of juice **now** or 3 squirts **in 10 minutes**.

**Typical choices**

Many people choose the burger / 2 squirts in the first case, and the salad / 3 squirts in the second.

**The same choice, delayed**

1. salad or burger for lunch **next Tuesday**;
2. 2 squirts of juice **in 10 minutes** or 3 squirts **in 20 minutes**.

A violation of stationarity: the decision maker would **revise on Tuesday** the plan she makes today.

# Temptation or changing tastes?

**Changing tastes?**

Preference reversals are not always evidence of temptation:

- tastes can change: you just had a salad, so you fancy a burger today but a salad next week;
- you are thirsty now, but expect not to be in 10 minutes.

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But such changes should go in **both directions** at random. In many cases they go **systematically** towards the immediate reward.

**Thirsty subjects**

| Choice | sooner | later (twice the amount) |
|---|---|---|
| now vs. in 5 minutes | 60% | 40% |
| in 20 vs. in 25 minutes | 30% | 70% |

# Exponential discounting rules out reversals

**Three dates: now, in 10 minutes, in 20 minutes**

If you prefer 2 squirts now to 3 squirts in 10 minutes:

$$u(2) + \delta u(0) \;\geq\; u(0) + \delta u(3) \quad \Longleftrightarrow \quad \delta \leq \frac{u(2) - u(0)}{u(3) - u(0)}.$$

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**Multiply by $\delta$**

Multiply both sides of the first inequality by $\delta > 0$:

$$\delta u(2) + \delta^2 u(0) \;\geq\; \delta u(0) + \delta^2 u(3),$$

so you also prefer 2 squirts in 10 minutes to 3 squirts in 20 minutes. More generally, for all $k$:

$$\delta^k u(2) + \delta^{k+1} u(0) \;\geq\; \delta^k u(0) + \delta^{k+1} u(3).$$

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**Time invariance:** if you prefer $x$ now to $y$ in $s$ periods, you prefer $x$ in $k$ periods to $y$ in $k + s$ periods. Exponential discounting cannot explain the data: **a new model is needed**.

3

# Quasi-hyperbolic discounting

One extra parameter, a present bias, explains preference reversals and keeps exponential discounting for the long run.

# The $\beta$–$\delta$ model

**Quasi-hyperbolic discounting**

A small change to exponential discounting, with big effects (Laibson, 1997):

$$D(0) = 1, \qquad D(\tau) = \beta\,\delta^{\tau} \quad \text{for } \tau \geq 1, \qquad 0 < \beta \leq 1,\ 0 < \delta \leq 1.$$

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## Viewed from date $t$

$$U_t(\mathbf{x}) = u(x_t) + \beta \sum_{\tau > t} \delta^{\tau - t} u(x_\tau).$$

## Interpretation

- $\beta < 1$: **present bias**. *All* future periods are discounted by an extra factor $\beta$ relative to today.
- $\beta = 1$: exponential discounting.
- Between two **future** dates, the ratio is $\delta$; between **today** and tomorrow, it is $\beta\delta$.

# Juice in the $\beta$–$\delta$ model

**Now vs. in 10 minutes**

Prefer 2 squirts **now** to 3 squirts in 10 minutes if

$$u(2) + \beta\delta u(0) \;\geq\; u(0) + \beta\delta u(3) \quad\Longleftrightarrow\quad \beta\delta \;\leq\; r \equiv \frac{u(2) - u(0)}{u(3) - u(0)}.$$

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**In 10 vs. in 20 minutes**

Prefer 3 squirts in **20** minutes to 2 squirts in **10** minutes if

$$\beta\delta u(2) + \beta\delta^2 u(0) \;\leq\; \beta\delta u(0) + \beta\delta^2 u(3) \quad\Longleftrightarrow\quad \delta \;\geq\; r.$$

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**Preference reversal** whenever

$$\beta\delta \;<\; r \;<\; \delta.$$

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**Example**

$u(k) = k$, so $r = \frac{2}{3}$. With $\delta = 0.95$ (per 10 minutes) and $\beta = 0.6$: $\beta\delta = 0.57 < 0.67 < 0.95$.

# Back to the opening poll

**The poll**

€100 today vs. €110 in one week; €100 in 52 weeks vs. €110 in 53 weeks. Linear $u$, weekly $\delta$.

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**Two decisions**

- **Today:** take €100 iff $100 \geq \beta\delta \cdot 110$.
- **In a year:** take €110 iff $\beta\delta^{53} \cdot 110 \geq \beta\delta^{52} \cdot 100$, i.e. iff $\delta \cdot 110 \geq 100$: $\beta$ cancels.

# Back to the opening poll

**The poll**

€100 today vs. €110 in one week; €100 in 52 weeks vs. €110 in 53 weeks. Linear $u$, weekly $\delta$.

**The reversal**

With $\delta \approx 1$ per week, the reversal "€100 now, €110 in 53 weeks" requires

$$\beta < \frac{100}{110} \approx 0.91.$$

**Two decisions**

- **Today:** take €100 iff $100 \geq \beta\delta \cdot 110$.
- **In a year:** take €110 iff $\beta\delta^{53} \cdot 110 \geq \beta\delta^{52} \cdot 100$, i.e. iff $\delta \cdot 110 \geq 100$: $\beta$ cancels.

One parameter, $\beta$, explains the reversal; $\delta$ governs the long-run trade-off.

# Recap: discounting

**Recap**

- Time consistency of plans $\iff$ **exponential** discounting (Strotz).
- Exponential discounting $\iff$ **stationarity** (with the other axioms; Koopmans).
- Systematic preference reversals towards the present **violate stationarity**.
- The $\beta$–$\delta$ **model** keeps $\delta$ for the long run and adds a present bias $\beta < 1$.

Next: what does a $\beta$–$\delta$ agent actually **do** when today's plan will be revised tomorrow?

# 4

# Consumption and savings with present bias

In a three-period savings problem, tomorrow’s self overturns today’s plan; naive and sophisticated agents alike would gain from commitment.

# A three-period savings problem

## Setup

- The decision maker lives for **3 periods** and earns income $I$ in each.
- In each period she consumes $x_t$; she can borrow and save at interest rate $0$, but cannot die in debt.
- She has $\beta$–$\delta$ preferences with CRRA utility

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \quad (\sigma > 0,\ \sigma \neq 1),$$

$$u(c) = \ln c \quad (\sigma = 1).$$

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$$u(c) = \ln c \quad (\sigma = 1).$$

## How much does she consume in each period?

Three scenarios:

1. **Commitment:** Self 1 chooses $x_1, x_2, x_3$ once and for all.
2. **Naive:** each self chooses its own consumption; Self 1 believes later selves will follow her plan.
3. **Sophisticated:** each self chooses its own consumption and anticipates later selves' choices.

# Commitment: Self 1's plan

**Self 1's problem**

$$\max_{x_1,x_2,x_3} \; u(x_1) + \beta\delta u(x_2) + \beta\delta^2 u(x_3) \quad \text{s.t.} \quad x_1 + x_2 + x_3 = 3I.$$

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**First-order conditions**

With multiplier $\lambda$, the first-order conditions are

$$u'(x_1) = \beta\delta\, u'(x_2) = \beta\delta^2\, u'(x_3) \;=\; \lambda.$$

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With CRRA, $u'(c) = c^{-\sigma}$:

$$x_1^{-\sigma} = \beta\delta\, x_2^{-\sigma} = \beta\delta^2 x_3^{-\sigma}$$

$$\implies \quad x_2 = (\beta\delta)^{1/\sigma} x_1, \quad x_3 = (\beta\delta^2)^{1/\sigma} x_1.$$

# Commitment: Self 1's plan

## Self 1's problem

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With CRRA, $u'(c) = c^{-\sigma}$:

$$x_1^{-\sigma} = \beta\delta\, x_2^{-\sigma} = \beta\delta^2 x_3^{-\sigma}$$

$$\Longrightarrow \quad x_2 = (\beta\delta)^{1/\sigma} x_1, \quad x_3 = (\beta\delta^2)^{1/\sigma} x_1.$$

## Solution

Plug into the budget constraint:

$$x_1^C = \frac{3I}{1 + (\beta\delta)^{1/\sigma} + (\beta\delta^2)^{1/\sigma}},$$

$$\boxed{\frac{x_3^C}{x_2^C} = \delta^{1/\sigma}}$$

# The intrapersonal game

Without commitment, each self chooses its **own** consumption. Let $w_t$ = wealth carried into period $t$ (so $w_2 = I - x_1$, possibly negative).

**Three selves**

- **Self 1** chooses $x_1$ (hence $w_2$) to maximize $u(x_1) + \beta\delta u(x_2) + \beta\delta^2 u(x_3)$.
- **Self 2** chooses $x_2$ (hence $w_3$) to maximize $u(x_2) + \beta\delta u(x_3)$.
- **Self 3** chooses $x_3$ to maximize $u(x_3)$.

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**Three selves**

- **Self 1** chooses $x_1$ (hence $w_2$) to maximize $u(x_1) + \beta\delta u(x_2) + \beta\delta^2 u(x_3)$.
- **Self 2** chooses $x_2$ (hence $w_3$) to maximize $u(x_2) + \beta\delta u(x_3)$.
- **Self 3** chooses $x_3$ to maximize $u(x_3)$.

**In Strotz's terms**

- **Consistent plan** = subgame-perfect equilibrium between the three selves: solve by **backward induction**.
- **Precommitment** = Self 1's plan is carried out (previous slide).

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**Self 2's choice**

FOC: $x_2^{-\sigma} = \beta\delta\, x_3^{-\sigma}$, so

$$x_2 = \frac{w_2 + 2I}{1 + (\beta\delta)^{1/\sigma}},$$

$$x_3 = (\beta\delta)^{1/\sigma}\, \frac{w_2 + 2I}{1 + (\beta\delta)^{1/\sigma}}.$$

$$\Longrightarrow \quad \boxed{\frac{x_3}{x_2} = (\beta\delta)^{1/\sigma}}$$

# Self 1 and Self 2 disagree

**Self 1's plan (commitment)**

$$\frac{x_3}{x_2} = \delta^{1/\sigma}$$

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If Self 1 could **commit** to a consumption stream, restricting Self 2's choice, she would want to: time inconsistency creates a **preference for commitment**, as in Gul and Pesendorfer (2001).

# The naive agent

**Self 1's choice**

Self 1 believes Self 2 will follow her plan, so she chooses

$$x_1^N = x_1^C = \frac{3I}{1 + (\beta\delta)^{1/\sigma} + (\beta\delta^2)^{1/\sigma}}.$$

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**Result**

Compared with the **commitment** plan, the naive agent

- consumes the **same** amount in period 1,
- consumes **more** in period 2,
- consumes **less** in period 3.

## The sophisticated agent

**Self 1's problem**

Self 1 knows that Self 2 will not use her savings as she would like. With $W = w_2 + 2I = 3I - x_1$ and $g \equiv (\beta\delta)^{1/\sigma}$, she chooses $x_1$ to maximize

$$u(x_1) + \beta\delta\, u\Big(\frac{W}{1+g}\Big) + \beta\delta^2\, u\Big(\frac{g\,W}{1+g}\Big), \qquad W = 3I - x_1.$$

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**Simplify**

Since CRRA is homogeneous, $u(aW) = a^{1-\sigma}u(W)$, this is $u(x_1) + K\,u(3I - x_1)$ with

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**Solution**

FOC: $x_1^{-\sigma} = K(3I - x_1)^{-\sigma}$, so

$$x_1^S = \frac{3I}{1 + K^{1/\sigma}}$$

vs. $$x_1^C = \frac{3I}{1 + (\beta\delta)^{1/\sigma} + (\beta\delta^2)^{1/\sigma}}.$$

## Which is higher: $x_1^S$ or $x_1^C$?

**Two opposing forces**

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- Self 3 is left with **too little**: a reason to **save more**, to protect Self 3.
- Which force wins depends on $\sigma$: with $\sigma > 1$ she saves more ($x_1^S < x_1^C$), with $\sigma < 1$ less; with $\sigma = 1$ (log) the two forces **cancel**: $x_1^S = x_1^C$.

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**In all cases**

The ratio $x_2/x_3$ is **higher** than Self 1 would like.

Sophistication does not solve the problem: Self 1 is still worse off than under commitment and would **pay** for a commitment device.

## Worked example: $\beta = \frac{1}{2}$, $\delta = 1$, $I = 40$, $u = \ln$

### Consumption paths

| | $x_1$ | $x_2$ | $x_3$ | $x_3/x_2$ | Self 1's utility |
|---|---|---|---|---|---|
| Commitment | 60 | 30 | 30 | 1 | $\ln 60 + \ln 30 \approx 7.496$ |
| Naive | 60 | 40 | 20 | $\frac{1}{2}$ | $\ln 60 + \frac{1}{2} \ln 800 \approx 7.437$ |
| Sophisticated | 60 | 40 | 20 | $\frac{1}{2}$ | $\approx 7.437$ |

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## Computations

- Commitment: $x_1 = 120/(1 + \frac{1}{2} + \frac{1}{2}) = 60$. Self 2 with $W = 60$: $x_2 = 60/(1 + \frac{1}{2}) = 40$.
- With $\sigma = 2$: $x^C = (49.7, 35.1, 35.1)$, $x^N = (49.7, 41.2, 29.1)$, $x^S = (49.3, 41.4, 29.3)$: the sophisticate saves more.

## Value of commitment

Self 1 would give up about **€3.5** of her €120 lifetime income to commit ($2\ln W - \ln 8 = 7.437 \Rightarrow W \approx 116.5$).

# Recap

**Discounting**

- Systematic **preference reversals** challenge exponential discounting: they violate **stationarity**.
- The $\beta$–$\delta$ **model** ($\beta < 1$: present bias) explains them with one extra parameter.

**Consumption and savings**

- With $\beta < 1$, today's plan is revised tomorrow (Strotz): Self 2 over-consumes relative to Self 3.
- **Naive** agents are surprised by it; **sophisticated** agents anticipate it but cannot prevent it.

Both would **gain** from commitment: time inconsistency is a foundation for **preference for commitment** (Lecture 17).

# Exercises

**1. (Easy)**

In the opening poll, with linear utility and weekly $\delta = 1$, for which $\beta$ does someone choose €100 today over €110 in one week, but €110 in 53 weeks over €100 in 52 weeks? What if $\delta = 0.99$?

**2. (Intermediate)**

In the three-period problem with $\beta = 1$, show that Self 2's choice coincides with Self 1's plan. Interpret in terms of Strotz's theorem.

**3. (Intermediate)**

In the worked example ($\beta = \frac{1}{2}$, $\delta = 1$, $I = 40$, log), suppose a bank offers a savings account that locks Self 1's plan, for a fee $F$ paid in period 1. Up to which $F$ does Self 1 accept? Does a naive Self 1 accept?

**4. (Advanced)**

Derive $x_1^S$ for CRRA utility. Show that $x_1^S = x_1^C$ when $\sigma = 1$, and check numerically that $x_1^S < x_1^C$ for $\sigma = 2$, $\beta = \frac{1}{2}$, $\delta = 1$, $I = 40$.

## Final take-aways

1. Only **exponential** discounting makes plans time-consistent; it is characterized by **stationarity**.
2. Systematic **preference reversals** towards the present reject stationarity.
3. The $\beta$–$\delta$ **model** adds present bias $\beta < 1$ and makes today's self disagree with tomorrow's.
4. **Naive** and **sophisticated** agents both under-save relative to their own plan; commitment is valuable.
5. Next lecture: is there evidence for present bias, for commitment, and for sophistication?

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