Solutions to Problem Set 02: Revealed Preference

Goes with Lecture 2. You test choice data for consistency (transitivity, WARP, GARP), prove small extensions of the representation theorems, and ask how much a passed or failed test tells us. **Solutions only; see the problem set for the statements.**

Notation (as in the lecture). X is a finite set of alternatives. $x \succ^\star y$: x is chosen from $\{x, y\}$. A utility function u *represents* $\succ^\star$ if $x \succ^\star y \iff u(x) > u(y)$ for all $x, y \in X$. $\mathcal{A}$ is the set of nonempty menus $A \subseteq X$, and $C : \mathcal{A} \to X$ is a choice function. For budget data, $\boldsymbol{x}_t \succsim^D \boldsymbol{x}_s$ iff $\boldsymbol{p}_t \cdot \boldsymbol{x}_t \geq \boldsymbol{p}_t \cdot \boldsymbol{x}_s$, and $\boldsymbol{x}_t \succ^D \boldsymbol{x}_s$ iff $\boldsymbol{p}_t \cdot \boldsymbol{x}_t > \boldsymbol{p}_t \cdot \boldsymbol{x}_s$.

Problem 1 Ordinal utility

Warm-up

Solution. (a) We need $v(\text{P}) > v(\text{B}) > v(\text{A})$. $v_1 = (100, 25, 4)$: yes ($t \mapsto t^2$ is strictly increasing for $t > 0$). $v_2 = (10, 15, 18)$: no, the ranking is reversed. $v_3 = (\ln 10, \ln 5, \ln 2) \approx (2.30, 1.61, 0.69)$: yes ($\ln$ is strictly increasing). $v_4 = (25, 0, 9)$: no, it ranks Avocado above Burger ($t \mapsto (t-5)^2$ is not increasing on $[2, 10]$). $v_5 = (5, 5, 2)$: no. $\min\{t, 5\}$ is increasing but not *strictly*; it gives $v_5(\text{P}) = v_5(\text{B})$, while P $\succ^\star$ B requires $v(\text{P}) > v(\text{B})$.

(b)(i) Each $r \in u(X)$ is the value of at least one alternative. For each $r \in u(X)$ pick one such alternative and call it $u^{-1}(r)$; then $u(u^{-1}(r)) = r$ by construction. (u need not be one-to-one, so u^{-1} is a selection, not an inverse: if $u(x) = u(x')$, we keep only one of them.) Since u represents $\succ^\star$, $u^{-1}(r) \succ^\star u^{-1}(s) \iff u(u^{-1}(r)) > u(u^{-1}(s)) \iff r > s$.

(ii) Let $r, s \in u(X)$ with $r > s$. By (i), $u^{-1}(r) \succ^\star u^{-1}(s)$. Since v represents $\succ^\star$, $v(u^{-1}(r)) > v(u^{-1}(s))$, i.e. $(v \circ u^{-1})(r) > (v \circ u^{-1})(s)$.

(iii) Let $f = v \circ u^{-1}$; it is strictly increasing on $u(X)$ by (ii). Take $x \in X$ and let $x' = u^{-1}(u(x))$. By (i), $u(x') = u(x)$. Since u represents $\succ^\star$, neither $x \succ^\star x'$ nor $x' \succ^\star x$. Since v represents $\succ^\star$ too, neither $v(x) > v(x')$ nor $v(x') > v(x)$, so $v(x) = v(x') = f(u(x))$. (Since $u(X)$ is finite, f can be extended to a strictly increasing function on $\mathbb{R}$, e.g. by linear interpolation between the points and slope 1 outside them.)

(c) Both statements must hold for every utility that represents the choices, since each one is as good as any other. Ratios: $u(\text{P})/u(\text{B}) = 2$, but $v_1(\text{P})/v_1(\text{B}) = 4$ and $v_3(\text{P})/v_3(\text{B}) = \ln 10 / \ln 5 \approx 1.43$. Differences: with u, $10 - 5 = 5 > 3 = 5 - 2$; with v_1, $75 > 21$; with v_3, $\ln 10 - \ln 5 = \ln 2 \approx 0.69 < \ln 5 - \ln 2 = \ln 2.5 \approx 0.92$. Neither the ratio nor the comparison of differences is the same across representations, so neither is revealed by choices. Only the ranking is: utility is ordinal.

Problem 2 Four dishes

Warm-up

Solution. (a) Complete: every pair was offered and exactly one dish chosen. Not transitive. The violations ($x \succ^\star y$, $y \succ^\star z$, but not $x \succ^\star z$) are exactly the three rotations of the cycle B $\succ^\star$ A $\succ^\star$ S $\succ^\star$ B: B $\succ^\star$ A, A $\succ^\star$ S but S $\succ^\star$ B; A $\succ^\star$ S, S $\succ^\star$ B but B $\succ^\star$ A; S $\succ^\star$ B, B $\succ^\star$ A but A $\succ^\star$ S. Any chain through P is fine: P beats everything and nothing beats P.

(b) $u(\text{P}) = 3$, $u(\text{B}) = 1$ (beats A), $u(\text{A}) = 1$ (beats S), $u(\text{S}) = 1$ (beats B). It does not represent $\succ^\star$: B $\succ^\star$ A but $u(\text{B}) = u(\text{A})$. The proof's step ($\Rightarrow$) fails: it uses transitivity to say that if $x \succ^\star y$ then x beats everything y beats. Here B beats A, but B does not beat S, which A beats.

(c) A representation requires transitivity, so the change must break the cycle B–A–S; changing a

choice that involves P leaves the cycle intact. Each of the three choices in the cycle works:

- A from $\{\mathsf{B},\mathsf{A}\}$: order P $\succ^\star$ A $\succ^\star$ S $\succ^\star$ B, $u = (\mathsf{P},\mathsf{A},\mathsf{S},\mathsf{B}) = (3,2,1,0)$.
- S from $\{\mathsf{A},\mathsf{S}\}$: order P $\succ^\star$ S $\succ^\star$ B $\succ^\star$ A, $u = (\mathsf{P},\mathsf{S},\mathsf{B},\mathsf{A}) = (3,2,1,0)$.
- B from $\{\mathsf{B},\mathsf{S}\}$: order P $\succ^\star$ B $\succ^\star$ A $\succ^\star$ S, $u = (\mathsf{P},\mathsf{B},\mathsf{A},\mathsf{S}) = (3,2,1,0)$.

In each case one checks all six pairs: the chosen dish has the higher utility. (These u are the counting functions of the new data.)

(d) The remaining choices are P $\succ^\star$ B, A, S, B $\succ^\star$ A, S $\succ^\star$ B. Not transitive: S $\succ^\star$ B and B $\succ^\star$ A, but not S $\succ^\star$ A (the pair was not offered). No representation: it would need $u(\mathsf{S}) > u(\mathsf{B}) > u(\mathsf{A})$, hence $u(\mathsf{S}) > u(\mathsf{A})$, hence S $\succ^\star$ A, which was not observed. As in the lecture, a representation reads a missing observation as a tie, and here a tie between S and A is impossible.

Problem 3 Choice functions and WARP

Core

Solution. (a) A choice function picks one element of each menu, so the number of choice functions is the product of the menu sizes. For $|X| = 3$: three singletons (1 option each), three pairs (2 each), one triple (3): $1^3 \cdot 2^3 \cdot 3 = 24$.

By the theorem, C satisfies WARP iff it picks the best element of every menu for a complete, transitive and asymmetric $\succ$ (a strict ranking of X). Different rankings give different choice functions: two rankings differ on some pair, and C on that pair reveals which one is used. So the number of WARP choice functions equals the number of strict rankings, $3! = 6$. The probability for a random chooser is $6/24 = \frac{1}{4}$.

For $|X| = 4$: $1^4 \cdot 2^6 \cdot 3^4 \cdot 4^1 = 64 \cdot 81 \cdot 4 = 20{,}736$ choice functions, of which $4! = 24$ satisfy WARP; probability $24/20{,}736 = 1/864 \approx 0.12\%$. With choices from all menus, WARP is a demanding test: a random chooser almost never passes it.

(b) $C(\{\mathsf{P},\mathsf{B}\}) = \mathsf{B}$, $C(\{\mathsf{B},\mathsf{A}\}) = \mathsf{A}$, $C(\{\mathsf{P},\mathsf{A}\}) = \mathsf{A}$, $C(\{\mathsf{P},\mathsf{B},\mathsf{A}\}) = \mathsf{B}$, and $C(\{w\}) = w$ for singletons. Binary choices: A $\succ^\star$ B, A $\succ^\star$ P, B $\succ^\star$ P: complete and transitive (the reverse of u). WARP fails: $C(\{\mathsf{P},\mathsf{B},\mathsf{A}\}) = \mathsf{B}$ with A available, and $C(\{\mathsf{B},\mathsf{A}\}) = \mathsf{A}$ with B available. So consistent binary choices do not imply WARP; this is the lecture's point that WARP links choices across menus.

(c) Let $A \subseteq B$, $x = C(B) \in A$, and suppose $C(A) = y \neq x$. Then $x, y \in A \cap B$ (as $y \in A \subseteq B$), $C(B) = x$ and $C(A) = y$. WARP applied to the menus B and A says: since $C(B) = x$, $C(A) \neq y$. Contradiction. So $C(A) = x$.

(d) Let $x \neq y$, both in $A \cap B$, with $C(A) = x$, and suppose $C(B) = y$. Apply contraction to $\{x, y\} \subseteq A$: since $C(A) = x \in \{x, y\}$, $C(\{x, y\}) = x$. Apply it to $\{x, y\} \subseteq B$: since $C(B) = y \in \{x, y\}$, $C(\{x, y\}) = y$. Contradiction, so $C(B) \neq y$: WARP holds. Hence, for choice functions observed on all menus, contraction and WARP are equivalent.

On Robinson's choices: $C(\{\mathsf{P},\mathsf{B},\mathsf{A}\}) = \mathsf{B} \in \{\mathsf{B},\mathsf{A}\}$, but $C(\{\mathsf{B},\mathsf{A}\}) = \mathsf{A} \neq \mathsf{B}$. Contraction fails, as it must.

Problem 4 Budget data: GARP and how much it tells us

Core

Solution. (a) Costs $\boldsymbol{p}_t \cdot \boldsymbol{x}_s$ (row t = prices of day t, column s = bundle of day s); the diagonal is spending:

	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$
$\boldsymbol{p}_1$	**12**	11	13
$\boldsymbol{p}_2$	22	**15**	14
$\boldsymbol{p}_3$	42	23	**16**

Day 1: $\boldsymbol{x}_2$ costs $11 < 12$: $\boldsymbol{x}_1 \succ^D \boldsymbol{x}_2$ (hence also $\succsim^D$); $\boldsymbol{x}_3$ costs $13 > 12$: no relation. Day 2: $\boldsymbol{x}_3$ costs $14 < 15$: $\boldsymbol{x}_2 \succ^D \boldsymbol{x}_3$; $\boldsymbol{x}_1$ costs $22 > 15$: none. Day 3: $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ cost 42 and 23, both > 16: none. (Each bundle is trivially $\succsim^D$ itself.)

(b) The only relations between different bundles are $\boldsymbol{x}_1 \succ^D \boldsymbol{x}_2 \succ^D \boldsymbol{x}_3$, so $\boldsymbol{x}_1 \succsim^R \boldsymbol{x}_2$, $\boldsymbol{x}_2 \succsim^R \boldsymbol{x}_3$, $\boldsymbol{x}_1 \succsim^R \boldsymbol{x}_3$. GARP would fail only if, in addition, $\boldsymbol{x}_2 \succ^D \boldsymbol{x}_1$, $\boldsymbol{x}_3 \succ^D \boldsymbol{x}_2$ or $\boldsymbol{x}_3 \succ^D \boldsymbol{x}_1$; none holds. GARP holds.

(c) Let u be locally non-satiated and rationalize the data. Since $\boldsymbol{p}_1 \cdot \boldsymbol{x}_2 = 11 < 12$, all bundles close enough to $\boldsymbol{x}_2$ cost less than 12 at $\boldsymbol{p}_1$; by local non-satiation one of them, $\boldsymbol{x}'$, has $u(\boldsymbol{x}') > u(\boldsymbol{x}_2)$. It is in B_1, so $u(\boldsymbol{x}_1) \geq u(\boldsymbol{x}') > u(\boldsymbol{x}_2)$. The same argument on day 2 ($\boldsymbol{p}_2 \cdot \boldsymbol{x}_3 = 14 < 15$) gives $u(\boldsymbol{x}_2) > u(\boldsymbol{x}_3)$. Hence $u(\boldsymbol{x}_1) > u(\boldsymbol{x}_3)$. Revealed preference is transitive through the chain, even where no direct comparison exists.

(d) New costs:

	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$
$\boldsymbol{p}_1$	**12**	11	11
$\boldsymbol{p}_2$	22	**15**	18
$\boldsymbol{p}_3$	14	18	**15**

Direct relations: $\boldsymbol{x}_1 \succ^D \boldsymbol{x}_2$, $\boldsymbol{x}_1 \succ^D \boldsymbol{x}_3$ (day 1), $\boldsymbol{x}_3 \succ^D \boldsymbol{x}_1$ (day 3, $14 < 15$); day 2 reveals nothing. So $\boldsymbol{x}_1 \succsim^R \boldsymbol{x}_3$ and $\boldsymbol{x}_3 \succ^D \boldsymbol{x}_1$: **GARP fails**.

Efficiency index. With efficiency e, the possible relations are: $\boldsymbol{x}_1$ over $\boldsymbol{x}_2$ and over $\boldsymbol{x}_3$ iff $12e \geq 11$, i.e. $e \geq \frac{11}{12}$ (strict iff $e > \frac{11}{12}$); $\boldsymbol{x}_3$ over $\boldsymbol{x}_1$ iff $15e \geq 14$, i.e. $e \geq \frac{14}{15}$ (strict iff $e > \frac{14}{15}$); nothing else for $e \leq 1$ (the other costs exceed spending). $\boldsymbol{x}_2$ is never revealed preferred to anything, so it lies on no cycle; a violation needs both $\boldsymbol{x}_1$ over $\boldsymbol{x}_3$ and $\boldsymbol{x}_3$ over $\boldsymbol{x}_1$, with one of them strict.

- $e < \frac{14}{15}$: $\boldsymbol{x}_3$ is not revealed preferred to $\boldsymbol{x}_1$; GARP$_e$ holds.
- $e = \frac{14}{15}$: $\boldsymbol{x}_3 \succsim_e^D \boldsymbol{x}_1$ (since $15e = 14$) and $\boldsymbol{x}_1 \succ_e^D \boldsymbol{x}_3$ (since $12e = 11.2 > 11$). GARP$_e$ fails. It fails also for every $e > \frac{14}{15}$.

The index is $\frac{14}{15} \approx 0.933$, and it is *not* attained: GARP$_e$ holds exactly for $e < \frac{14}{15}$. Interpretation: on day 3 Robinson paid 15 for $\boldsymbol{x}_3$, while $\boldsymbol{x}_1$, which he revealed on day 1 to be strictly better, cost only 14; he wasted $1/15 \approx 6.7\%$ of his day-3 budget.

Lecture example: spending is 9 on both days and the other bundle costs 6, so each bundle is revealed preferred to the other iff $9e \geq 6$, strictly iff $9e > 6$. For $e < \frac{2}{3}$ there are no relations; at $e = \frac{2}{3}$ both relations are weak and none is strict, so GARP$_e$ holds; for $e > \frac{2}{3}$ both are strict and GARP$_e$ fails. The index is $\frac{2}{3}$, and here it *is* attained ("one third of income wasted"). Strictly speaking the index is a supremum; the "largest" e exists only in cases like this one.

(e) If $\boldsymbol{p}_t = \boldsymbol{p}$ for all t, then $\boldsymbol{x}_t \succsim^D \boldsymbol{x}_s$ iff $\boldsymbol{p} \cdot \boldsymbol{x}_t \geq \boldsymbol{p} \cdot \boldsymbol{x}_s$. Along any chain $\boldsymbol{x}_t \succsim^D \boldsymbol{x}_{t_1} \succsim^D \cdots \succsim^D \boldsymbol{x}_s$ spending weakly falls, so $\boldsymbol{x}_t \succsim^R \boldsymbol{x}_s$ implies $\boldsymbol{p} \cdot \boldsymbol{x}_t \geq \boldsymbol{p} \cdot \boldsymbol{x}_s$. But $\boldsymbol{x}_s \succ^D \boldsymbol{x}_t$ would mean $\boldsymbol{p} \cdot \boldsymbol{x}_s > \boldsymbol{p} \cdot \boldsymbol{x}_t$. So no violation is possible. (The same holds if all price vectors are proportional, $\boldsymbol{p}_t = \alpha_t \boldsymbol{p}$ with $\alpha_t > 0$, because the relations do not change when a row is multiplied by α_t.) The budget lines are parallel and never cross; the test has no power.

(f) GARP can only fail when budget lines cross, so that each chosen bundle can be compared with bundles on other lines. Budget lines with many different slopes make many such comparisons

possible, so a random or careless chooser would very likely fail; passing then means something. A high efficiency index shows that the choices are *close* to the choices of some utility maximizer: little money is wasted by inconsistent choices. It does not show which utility function the person has, that her choices are good for her (welfare), or that she would be consistent in other settings (other goods, menus, framings); and its meaning depends on the power of the design, which one judges by comparing with simulated random choosers on the same budgets.

Problem 5 Just noticeable differences

Core

Solution. (a) Asymmetric: $m - n > 100$ and $n - m > 100$ cannot both hold. Transitive: if $m - n > 100$ and $k - m > 100$ then $k - n > 200 > 100$, so $c_n \succ c_m \succ c_k$ implies $c_n \succ c_k$.

(b) $c_n \sim c_m$ iff $|n - m| \leq 100$. So $c_n \sim c_{n+1}$ for every n, but $c_0 \succ c_{1000}$ ($1000 > 100$). Smaller example: $c_0 \sim c_{100}$ and $c_{100} \sim c_{200}$, but $c_0 \succ c_{200}$, not $c_0 \sim c_{200}$.

(c) Complete: for any n, m, at most one of $c_n \succ c_m$, $c_m \succ c_n$ holds (asymmetry), so $c_n \succsim c_m$ or $c_m \succsim c_n$. Not transitive: $c_{200} \succsim c_{100}$ (since not $c_{100} \succ c_{200}$: the difference is exactly 100), $c_{100} \succsim c_0$, but not $c_{200} \succsim c_0$, because $c_0 \succ c_{200}$. By the lecture's weak-order theorem, a relation on a finite set is represented by some v (with $\geq$) iff it is complete and transitive; $\succsim$ is not transitive, so no such v exists. Directly: $v(c_{200}) \geq v(c_{100}) \geq v(c_0)$ would give $v(c_{200}) \geq v(c_0)$, i.e. $c_{200} \succsim c_0$, which is false.

(d) The lecture's pump needs a cycle of *strict* preferences, so that Robinson pays at every step. Here $\succ$ is transitive and has no cycle; the cycle runs through indifferences. Robinson would pay to move from c_{200} to c_0, but he has no strict reason to accept the steps $c_0 \to c_{100} \to c_{200}$, and he would not pay for them. A pump works only if he accepts trades about which he is indifferent, and any small payment to make him accept them is a cost to the pumper. So his choices violate transitivity of $\succsim$ without being easy to exploit. The intransitivity comes from a perception limit, not from inconsistent tastes; whether this is "irrational" depends on whether perfect perception is part of rationality. This supports the lecture's point that violating transitivity is not always irrational.

Problem 6 Counting cycles: a degree of inconsistency

Challenge

Solution. (a) Each of the $\binom{n}{2} = n(n-1)/2$ pairs is offered once and has exactly one winner, which gets one point. So the scores add up to $n(n-1)/2$.

(b)(i) If $\succ^\star$ is transitive, the lecture's theorem shows that u represents $\succ^\star$. By completeness, $x \neq y$ implies $x \succ^\star y$ or $y \succ^\star x$, so $u(x) \neq u(y)$. Each score lies in $\{0, \ldots, n-1\}$ (an alternative beats at most the $n-1$ others), so the n distinct scores take each of these n values exactly once.

(ii) h has score $n-1$, so it beats all $n-1$ other alternatives. By asymmetry, nothing beats h. The score of $x \neq h$ counts the alternatives x beats; h is not among them, so removing h leaves every other score unchanged. The remaining scores are $0, \ldots, n-1$ without $n-1$, i.e. $0, \ldots, n-2$.

(iii) Induction on n. For $n \leq 2$ there is nothing to show. Assume the claim for $n-1$ alternatives. By (ii) and the induction hypothesis, $\succ^\star$ is transitive on $X \setminus \{h\}$. A chain $x \succ^\star y \succ^\star z$ that involves h must start with $x = h$ (nothing beats h), and then $h \succ^\star z$ holds. So $\succ^\star$ is transitive on X.

(c)(i) If $x \succ^\star y \succ^\star z \succ^\star x$, then $x \succ^\star y$ and $y \succ^\star z$, but $z \succ^\star x$, so not $x \succ^\star z$ by asymmetry: transitivity fails.

(ii) By asymmetry $x \neq y$ and $y \neq z$; and $x \neq z$, otherwise $x \succ^\star y \succ^\star x$. By completeness, not $x \succ^\star z$ gives $z \succ^\star x$, so $\{x, y, z\}$ is cyclic. With (i): $\succ^\star$ is transitive iff no triple is cyclic.

(iii) Inside a triple, three pairs are decided, so the three within-triple scores add up to 3, each

between 0 and 2. If some element scores 2, it beats the other two, and it is unique (two such elements would beat each other). Otherwise all three scores are at most 1 and add up to 3, so each element beats exactly one other: the triple is cyclic.

(iv) x beats $u(x)$ alternatives, and any two of them form a triple with x in which x beats both: $\binom{u(x)}{2}$ triples. By (iii), each non-cyclic triple has exactly one element that beats the other two, and a cyclic triple has none. So summing over x counts each non-cyclic triple exactly once: $\sum_x \binom{u(x)}{2}$ is the number of non-cyclic triples, and the number of cyclic triples is $\binom{n}{3} - \sum_x \binom{u(x)}{2}$.

(d)(i) Problem 2: scores $(3, 1, 1, 1)$, so $\binom{4}{3} - \left[\binom{3}{2} + 0 + 0 + 0\right] = 4 - 3 = 1$ cyclic triple, namely $\{\mathsf{B}, \mathsf{A}, \mathsf{S}\}$.

(ii) Replacing s by $s - 1$ changes $\binom{s}{2}$ by $-(s-1)$; replacing t by $t + 1$ changes $\binom{t}{2}$ by $+t$. The total change is $t - (s - 1) \leq -1$ when $s \geq t + 2$. Start from any scores (nonnegative integers adding up to 10). As long as two scores differ by 2 or more, such a move lowers $\sum_x \binom{u(x)}{2}$ and keeps the total at 10; it also lowers $\sum_x u(x)^2$, so the process stops. It stops when all scores are within 1 of each other; five integers adding up to 10 that are within 1 of each other all equal 2. So $\sum_x \binom{u(x)}{2} \geq 5\binom{2}{2} = 5$, and at most $\binom{5}{3} - 5 = 10 - 5 = 5$ triples are cyclic.

(iii) Two alternatives are 1 or 2 steps apart in one direction and 4 or 3 steps apart in the other, so exactly one of them beats the other: every pair is decided exactly once. Each x_i beats exactly x_{i+1} and x_{i+2}, so every score equals 2. By (c), the number of cyclic triples is $10 - 5 = 5$, the maximum from (ii). Example: $\{x_0, x_1, x_3\}$, with $x_0 \succ^\star x_1 \succ^\star x_3 \succ^\star x_0$.

(e) Advantage: it needs only the binary choices, is easy to compute from the scores, is 0 exactly for consistent choices (by (c)), and grades inconsistency instead of a pass/fail verdict (like Afriat's efficiency index for budget data). Weakness (one is enough): it counts every cycle the same, whatever the stakes; it says nothing about how many choices one must change to restore consistency; it depends on the number of alternatives offered, so it must be normalized to compare subjects; and with noisy choices between nearly equal options (Problem 5) it may record cycles that reflect perception, not inconsistent tastes.

References

Choi, S., Kariv, S., Müller, W., and Silverman, D. (2014). Who is (more) rational? *American Economic Review*, 104(6):1518–1550.