Problem Set 02: Revealed Preference

Goes with Lecture 2. You test choice data for consistency (transitivity, WARP, GARP), prove small extensions of the representation theorems, and ask how much a passed or failed test tells us.

Notation (as in the lecture). X is a finite set of alternatives. $x \succ^\star y$: x is chosen from $\{x, y\}$. A utility function u *represents* $\succ^\star$ if $x \succ^\star y \iff u(x) > u(y)$ for all $x, y \in X$. $\mathcal{A}$ is the set of nonempty menus $A \subseteq X$, and $C : \mathcal{A} \to X$ is a choice function. For budget data, $\boldsymbol{x}_t \succsim^D \boldsymbol{x}_s$ iff $\boldsymbol{p}_t \cdot \boldsymbol{x}_t \geq \boldsymbol{p}_t \cdot \boldsymbol{x}_s$, and $\boldsymbol{x}_t \succ^D \boldsymbol{x}_s$ iff $\boldsymbol{p}_t \cdot \boldsymbol{x}_t > \boldsymbol{p}_t \cdot \boldsymbol{x}_s$.

Problem 1 Ordinal utility

Warm-up

Robinson chooses Pizza over Burger, Burger over Avocado and Pizza over Avocado. The utility $u(\mathsf{P}) = 10$, $u(\mathsf{B}) = 5$, $u(\mathsf{A}) = 2$ represents these choices.

(a) Which of the following functions also represent them? Justify each answer.

$$v_1 = u^2, \qquad v_2 = 20 - u, \qquad v_3 = \ln u, \qquad v_4 = (u-5)^2, \qquad v_5 = \min\{u, 5\}.$$

(b) The lecture shows: if u represents $\succ^\star$, so does $f \circ u$ for every strictly increasing f. Prove a converse: if u and v both represent the same $\succ^\star$ on a finite X, then $v(x) = f(u(x))$ for all $x \in X$, for some function f that is strictly increasing on the set $u(X)$ of values of u. Proceed in three steps.

(i) Prove that there is a function $u^{-1} : u(X) \to X$ such that $u(u^{-1}(r)) = r$ for all $r \in u(X)$, and $u^{-1}(r) \succ^\star u^{-1}(s) \iff r > s$ for all $r, s \in u(X)$.

(ii) Prove that the function $v \circ u^{-1} : u(X) \to v(X)$ is strictly increasing.

(iii) Conclude: take $f = v \circ u^{-1}$ and show that $v(x) = f(u(x))$ for all $x \in X$.

(c) A student says: "Since $u(\mathsf{P}) = 2u(\mathsf{B})$, Pizza is twice as good as Burger." Another says: "Going from Burger to Pizza is a bigger improvement than going from Avocado to Burger, since $10 - 5 > 5 - 2$." Use (a) to show that neither statement has a meaning in this theory.

Problem 2 Four dishes

Warm-up

The canteen adds Sushi (S). Robinson is offered every pair once and chooses:

Menu	{P, B}	{P, A}	{P, S}	{B, A}	{A, S}	{B, S}
Chosen	P	P	P	B	A	S

(a) Is $\succ^\star$ complete? Is it transitive? List every violation of transitivity.

(b) Compute the lecture's counting function $u(x) = \#\{z \in X : x \succ^\star z\}$. Does it represent $\succ^\star$? Which step of the lecture's proof fails?

(c) Find *every* single choice in the table whose change makes the data representable by a utility function. For each, give a utility function.

(d) Go back to the original table, but suppose the pair $\{\mathsf{A}, \mathsf{S}\}$ was never offered. Is the remaining $\succ^\star$ transitive? Can a utility function represent it?

Problem 3 Choice functions and WARP

Core

(a) Let $X = \{x, y, z\}$. How many choice functions $C : \mathcal{A} \to X$ are there? How many satisfy WARP? (Use the lecture's theorem.) If a subject picked an option at random from each menu (uniformly and independently), what is the probability that her choices satisfy WARP? Answer the same questions for $|X| = 4$.

(b) Robinson ranks the dishes P, B, A by a utility u with $u(\text{P}) > u(\text{B}) > u(\text{A})$, but from every menu with at least two dishes he modestly takes the *second best*. Write down his choice function. Are his binary choices complete and transitive? Does C satisfy WARP?

(c) *Contraction.* Show that if C satisfies WARP, then for all menus $A \subseteq B$: if $C(B) \in A$, then $C(A) = C(B)$.

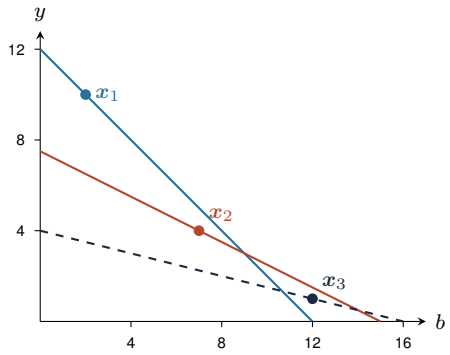
(d) Show the converse: if C satisfies the contraction property in (c), it satisfies WARP. Then check the contraction property directly on Robinson's choice function from (b).

Problem 4 Budget data: GARP and how much it tells us

Core

Robinson buys bourbon and yoga classes on three days:

Day t	Prices $\boldsymbol{p}_t$	Bundle $\boldsymbol{x}_t = (b_t, y_t)$
1	$(1, 1)$	$(2, 10)$
2	$(1, 2)$	$(7, 4)$
3	$(1, 4)$	$(12, 1)$



(a) Compute all costs $\boldsymbol{p}_t \cdot \boldsymbol{x}_s$ and list all direct revealed preferences, weak and strict.

(b) Do the data satisfy GARP?

(c) Show that every locally non-satiated utility u that rationalizes the data satisfies $u(\boldsymbol{x}_1) > u(\boldsymbol{x}_3)$, although $\boldsymbol{x}_3$ was not affordable on day 1.

(d) Suppose instead that on day 3 the prices were $\boldsymbol{p}_3 = (2, 1)$ and Robinson bought $\boldsymbol{x}_3 = (4, 7)$ (days 1 and 2 unchanged). Does GARP hold? For $e \in [0, 1]$, let GARP$_e$ be GARP with the relations $\boldsymbol{x}_t \succsim_e^D \boldsymbol{x}_s$ iff $e\,\boldsymbol{p}_t \cdot \boldsymbol{x}_t \geq \boldsymbol{p}_t \cdot \boldsymbol{x}_s$ and $\boldsymbol{x}_t \succ_e^D \boldsymbol{x}_s$ iff $e\,\boldsymbol{p}_t \cdot \boldsymbol{x}_t > \boldsymbol{p}_t \cdot \boldsymbol{x}_s$. Afriat's efficiency index is the supremum of the e for which GARP$_e$ holds (the lecture calls it the "largest" such e). Compute it. Compute it also for the lecture's worked example ($\boldsymbol{p}_1 = (1, 2)$, $\boldsymbol{x}_1 = (1, 4)$; $\boldsymbol{p}_2 = (2, 1)$, $\boldsymbol{x}_2 = (4, 1)$). Is the supremum attained in each case?

(e) Show that if prices never change ($\boldsymbol{p}_t = \boldsymbol{p}$ for all t), every dataset satisfies GARP, whatever Robinson buys.

(f) *Interpretation.* Choi et al. (2014) find that most subjects in budget experiments are close to consistent. Using (e), explain why such experiments draw budget lines with many different slopes. What does a high efficiency index show, and what does it not show?

Problem 5 Just noticeable differences

Core

Coffee c_n contains n grains of sugar, $n = 0, 1, \ldots, 1000$. Robinson likes less sugar, but he notices a difference only beyond 100 grains: let $u(c_n) = -n/100$ and

$$c_n \succ c_m \iff u(c_n) > u(c_m) + 1 \iff m - n > 100,$$

and $c_n \sim c_m$ iff neither $c_n \succ c_m$ nor $c_m \succ c_n$.

(a) Show that $\succ$ is asymmetric and transitive.

(b) Show that $\sim$ is not transitive. Check the lecture's claim: $c_0 \sim c_1 \sim \cdots \sim c_{1000}$, yet $c_0 \succ c_{1000}$.

(c) Let $c_n \succsim c_m$ iff not $c_m \succ c_n$. Show that $\succsim$ is complete but not transitive. Conclude that no utility function v satisfies $c_n \succsim c_m \iff v(c_n) \geq v(c_m)$.

(d) Can the lecture's money pump be run against Robinson? Is he irrational?

Problem 6 Counting cycles: a degree of inconsistency

Challenge

Let $|X| = n \geq 3$, and suppose every pair is offered once and exactly one option is chosen, so $\succ^\star$ is complete and asymmetric. Let $u(x) = \#\{z \in X : x \succ^\star z\}$ be the number of options x beats (its *score*).

(a) Show that $\sum_{x \in X} u(x) = n(n-1)/2$.

(b) Show that $\succ^\star$ is transitive iff the scores are exactly $0, 1, \ldots, n-1$ (each value taken once). Proceed in three steps.

(i) Suppose $\succ^\star$ is transitive. Use the lecture's theorem to show that two different alternatives never have the same score. Why does this force the scores to be exactly $0, 1, \ldots, n-1$?

(ii) Now suppose the scores are $0, 1, \ldots, n-1$, and let h be the alternative with score $n-1$. Show that h beats every other alternative and that nothing beats h. Show that if we remove h, the scores on $X \setminus \{h\}$ are $0, 1, \ldots, n-2$.

(iii) Conclude by induction on n: if the scores are $0, 1, \ldots, n-1$, then $\succ^\star$ is transitive. (Which chains $x \succ^\star y \succ^\star z$ can involve h?)

(c) Call a set $\{x, y, z\}$ of three alternatives *cyclic* if $x \succ^\star y \succ^\star z \succ^\star x$ (in some order). Show that $\succ^\star$ is transitive iff no triple is cyclic, and that

$$\#\{\text{cyclic triples}\} = \binom{n}{3} - \sum_{x \in X} \binom{u(x)}{2}.$$

Proceed in four steps.

(i) Show that if some triple is cyclic, $\succ^\star$ is not transitive.

(ii) Suppose $x \succ^\star y$ and $y \succ^\star z$ but not $x \succ^\star z$. Show that x, y, z are three different alternatives and that $\{x, y, z\}$ is cyclic. Conclude that $\succ^\star$ is transitive iff no triple is cyclic.

(iii) Take any triple. Show that either it is cyclic, or exactly one of its elements beats the other two. (Hint: inside the triple, three pairs are decided. How many points do the three elements score inside the triple?)

(iv) Fix x. How many triples $\{x, y, z\}$ are there in which x beats both y and z? Using (iii), deduce that $\sum_{x \in X} \binom{u(x)}{2}$ is the number of triples that are *not* cyclic, and conclude.

(d) Apply the formula to Problem 2. Then let $n = 5$. Show that at most 5 of the 10 triples can be cyclic, and give choices for which exactly 5 are. Proceed in three steps.

(i) Compute the scores in Problem 2 and the number of cyclic triples. Which triple is it?

(ii) Let $n = 5$. By (a), the scores add up to 10. Show that if two scores satisfy $s \geq t + 2$, replacing them by $s - 1$ and $t + 1$ lowers $\sum_x \binom{u(x)}{2}$. Deduce that $\sum_x \binom{u(x)}{2} \geq 5$, with the minimum when all scores equal 2, and hence that at most 5 triples are cyclic.

(iii) Let $X = \{x_0, \ldots, x_4\}$, where x_i beats x_{i+1} and x_{i+2} (indices mod 5). Check that every pair is decided exactly once and that every score is 2. How many triples are cyclic? Give one.

(e) The number of cyclic triples measures *how far* binary choices are from utility maximization. Give one advantage and one weakness of this measure.

References

Choi, S., Kariv, S., Müller, W., and Silverman, D. (2014). Who is (more) rational? *American Economic Review*, 104(6):1518–1550.