INTRODUCTION TO ECONOMICS FOR PSYCHOLOGISTS: HISTORY, CONCEPTS, AND PSYCHOLOGICAL FOUNDATIONS

PART 4

# Taming Chance: Choice under Uncertainty from Pascal to Prospect Theory

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# The weight of uncertainty



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How can we make sense of our concept of **utility** when there is **uncertainty**?

## Outline

Games of chance and the birth of frequentist probability

Probability as logic

Subjective probability (with objective utility)

Lotteries, objective probability and expected utility

Savage: subjective probability (and utility)

Experimental challenges to expected utility

Theories to account for the anomalies

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# Games of chance and the birth of frequentist probability

Gamblers' puzzles lead Pascal, Fermat, Bayes and the Bernoullis to the concepts of probability, expectation and utility.

# The problem of points

**The game**

Two players each stake 32 pistoles (gold coins) and toss a coin. Player 1 bets on **heads**, player 2 on **tails**. The game stops as soon as heads or tails has come up **3 times**; the winner takes all 64 pistoles.

**The problem**

- So far: **two tails and one head**.
- But for some reason, the game must stop **right now**.
- How should the 64 pistoles be split **fairly**, taking into account each player's successes so far?

## The problem of points: a 150-year puzzle

- The problem, raised by the **Chevalier de Méré** in the 17th century, kept mathematicians busy for about 150 years.
- The long-accepted answer: player 2 should get **two thirds** of the stakes (so $42.66\ldots$ pistoles), since she won 2/3 of the rounds played.
- **Pascal** and **Fermat** solve the problem and show that this answer is wrong.

In doing so, they open the way to a new mathematical theory: **probability**.

# De Méré's "scandal": a letter from Pascal to Fermat (1654)

*Sir, I did not have time to send you the proof of a difficulty that greatly astonished M. de Méré, for he is very clever, but he is no geometer (which, as you know, is a great defect) [. . . ] He told me that he had found a falsehood in numbers, for this reason:*
*If one undertakes to throw a six with one die, there is an advantage in undertaking it in 4 throws, as 671 to 625.*
*If one undertakes to throw a double six with two dice, there is a disadvantage in undertaking it in 24 throws.*
*And yet 24 is to 36 (the number of faces of two dice) as 4 is to 6 (the number of faces of one die).*
*That was his great scandal, which made him say loudly that propositions were not constant and that arithmetic contradicts itself; but you will easily see the reason with the principles you hold.*

Check: $P(\text{a six in 4 throws}) = 1 - (5/6)^4 \approx 0.518$; $P(\text{a double six in 24 throws}) = 1 - (35/36)^{24} \approx 0.491$.

# Pascal's solution

**If the game had gone on, two cases**

- With probability $\frac{1}{2}$: **tails**. Player 2 wins the 64 pistoles.
- With probability $\frac{1}{2}$: **heads**. The players are level, so it is fair that each takes back her stake: 32 each.

Player 2's expectation $= \frac{1}{2}(64+32) = \mathbf{48} \neq 42.66$

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**Pascal thus invents**

- the concept of **probability**;
- the concept of **expectation** (expected value).

The problem also has an **economic** side, much debated at the time: how to share wealth when a **contract** is broken off.

# Bayes' theory

- **Thomas Bayes**, a Presbyterian minister, sought to rebut the skepticism of his compatriot **Hume** about miracles!
- He invents the concept of **conditional probability**: the probability that an event occurs *given that* another event has occurred.

**Example**

- A population with 50% women and 50% men.
- 80% of women voted Macron (vs. Le Pen), against 50% of men.
- Pick someone at random. *Given that* this person voted Macron, what is the probability that she is a woman?

$$P(W \mid M) = \frac{P(M \mid W)\,P(W)}{P(M \mid W)\,P(W) + P(M \mid Men)\,P(Men)} = \frac{0.8 \times 0.5}{0.8 \times 0.5 + 0.5 \times 0.5} \approx \mathbf{62\%}$$

# The St. Petersburg paradox

**Nicolas Bernoulli** imagines a game that, ahead of its time, leads to the notions of **utility** and **expected utility**.

**The game**

- A player pays an entry fee, collected by a bank.
- The player tosses a coin as long as it lands **heads**.
- The game ends when **tails** first appears; the bank then pays the player:
  - $2^0 = 1$ € if tails first appears on the 1st toss;
  - $2^1 = 2$ € if tails first appears on the 2nd toss;
  - $2^2 = 4$ € if tails first appears on the 3rd toss; and so on.

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Would you play for an entry fee of **5 €**? **100 €**?

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Would you play for an entry fee of **5 €**? **100 €**? **One million €**?

# The St. Petersburg paradox: resolution

**The paradox**

If you only cared about your **expected gain**, you should *always* play, whatever the fee. Why? Because your expected gain is

$$\frac{1}{2} \cdot 1 + \frac{1}{2^2} \cdot 2 + \frac{1}{2^3} \cdot 2^2 + \cdots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = +\infty.$$

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Potential resolution: diminishing **marginal utility** of gains? **No: risk aversion!**

- Instead of valuing the gain $2^{n-1}$ as such, value it by its **utility** $\ln(2^{n-1})$:

$$\frac{\ln 1}{2} + \frac{\ln 2}{2^2} + \frac{\ln(2^2)}{2^3} + \frac{\ln(2^3)}{2^4} + \cdots = \ln 2 \left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots\right) = \ln 2 < +\infty.$$

The game is then worth a sure gain of only $2\,€$ (since $\ln 2$ is the utility of $2\,€$).

# The frequentist interpretation

**Frequentist probability**

The probability of an event is its **frequency**: the number of cases in which it occurs, divided by the total number of cases.

**Example.** The probability that a die lands on an even number is

$$\frac{\text{number of even faces}}{\text{number of faces}} = \frac{3}{6} = \frac{1}{2}.$$

**The law of large numbers**

When we draw at random from a large population, the larger the sample, the closer its statistical characteristics get to those of the population.

**Example.** The more often I roll the die, the closer the average of the faces gets to the expectation $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$.

## Probability in the early 20th century

The modern concept of probability takes shape, driven by mathematicians such as **Kolmogorov** and **Borel**:

- the correlation coefficient (Galton and Pearson);
- Kolmogorov's **axioms**, which give the concept rigorous foundations;
- the law of large numbers.

What interests us: this revolution also sparks **philosophical debates** on what probability *means*, between the **logical** interpretation and the **subjective** interpretation of probability.

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# Probability as logic

Keynes: probability as a rational degree of belief, logically linking evidence to conclusions — and the limits of numbers.

## Criticisms of frequentism

The frequentist interpretation is natural and intuitive, but it has grey areas.

**Lack of generality**

Sometimes we want to talk about probability, but frequentist logic does not allow it.

- It seems more likely that Jon Snow and Daenerys Targaryen kiss than that Tyrion and Tywin Lannister do;
- but there is no frequency, as there is for dice rolls: there is only one first and last version of *Game of Thrones*.

**Pragmatically limited**

There are many statistical tests and methods. The frequentist cannot say how and why a statistician should choose one rather than another.

# The logical conception of probability

Key reference: **J. M. Keynes**, *A Treatise on Probability* (1921).

> *"Part of our knowledge we obtain direct; and part by argument. The Theory of Probability is concerned with that part which we obtain by argument, and it treats of the different degrees in which the results so obtained are conclusive or inconclusive."*
>
> — Keynes (1921)

- A proposition can be true with some probability: "$a$ is true with probability $p$".
- So "$a$ is true" $=$ "$a$ is true with probability 1".
- **Probability $=$ rational degree of belief.**

# Probability versus the weight of arguments

**Keynes's distinction**

- **Probability**: how likely a proposition is to be true, compared with its contrary.
- **Weight of arguments**: how much information I have about whether the proposition is true.

**When I learn more about a proposition. . .**

- its **probability** may go *down*: I learn that my candidate is slumping in the polls (lower chance that she wins);
- but the **weight** of the arguments goes *up*, because I know more: I am better informed about the probability that my candidate wins.

## The logical conception and radical uncertainty

**An example by Georgescu-Roegen.** Four statements about an urn $U$:

E1. Urn $U$ contains 1/3 red balls and 2/3 black balls.
E2. Out of 3 426 000 draws with replacement from urn $U$, 1/3 gave red balls and 2/3 black balls.
E3. Out of 10 draws with replacement from urn $U$, 1/3 gave red balls and 2/3 black balls.
E4. Urn $U$ contains balls.

The probability of drawing red may look the same in E1–E3, but the **weight** of the evidence differs; in E4, there is nothing to base a probability on.

# Radical uncertainty: uncertainty without frequencies

Keynes's conception also suggests distinguishing uncertain propositions or events:

- those for which a **frequency** gives the probability;
- those for which there are **no frequencies**.

**Frank Knight (1921): risk vs. uncertainty**

The same distinction appears in Knight: we speak of **risk** (measurable probabilities) versus **radical uncertainty** (no probability can be measured).

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# Subjective probability (with objective utility)

Ramsey and de Finetti: probabilities live in people's heads, and we read them from the bets people accept.

# The subjective conception of probability

> *"[T]here really do not seem to be any such things as the probability relations [Keynes] describes. He supposes that, at any rate in certain cases, they can be perceived; but speaking for myself I feel confident that this is not true. I do not perceive them… moreover I shrewdly suspect that others do not perceive them either, because they are able to come to so very little agreement as to which of them relates any two given propositions."*
>
> — F. P. Ramsey, "Truth and Probability" (1926)

- **Agreement with Keynes:** probabilities are to be read as **beliefs**,
- **but** they are not objective beliefs: they are **in people's heads**.

## Reading beliefs from actions

**Problem:** how do I know what is in people's heads? **Answer:** I observe their **choices**.

> *"The degree of a belief is a causal property of it, which we can express vaguely as the extent to which we are prepared to act on it."*
>
> — F. P. Ramsey (1926)

Link with **pragmatist** philosophy (Peirce, James, Dewey): beliefs are defined not, as in Descartes, as ideas of the mind, but by **habits of action**, i.e. by the effects they produce on behavior.

> *"Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object."*
>
> — C. S. Peirce (1878)

## Measuring beliefs with bets

First contributors to this approach: **de Finetti** and **Ramsey**.

**Bets**

I want to measure the probability you attach to “Jon Snow and Daenerys Targaryen kiss”.

- I ask you to choose between
  1. $p_1$: I give you 1 € whatever happens;
  2. $p_2$: I give you $x$ € if they kiss, 0 otherwise.
- I then vary $x$ until you are **indifferent** between $p_1$ and $p_2$. This gives a number $x^*$, the **certainty equivalent** of the bet.

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Assuming that your utility for $x$ € is $x$ and that you maximize expected utility:

$$1 = p\,x^* + (1-p)\cdot 0 \quad \Longrightarrow \quad p\,x^* = 1 \quad \Longrightarrow \quad p = \frac{1}{x^*}$$

where $p$ is the probability you attach to the event "they kiss".

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# Lotteries, objective probability and expected utility

Von Neumann and Morgenstern take probabilities as given and derive utility from choices between lotteries.

# Expected utility à la von Neumann and Morgenstern

In *Theory of Games and Economic Behavior* (1944), von Neumann and Morgenstern do, in a sense, the **opposite**: probabilities are given, utility is to be found.

## Lotteries

Suppose there are $n$ possible sums of money to win, $x_1, \ldots, x_n$. A **lottery** $l = (p_1, \ldots, p_n)$ assigns to each sum $x_i$ a probability $p_i$, with

$$p_1 + \cdots + p_n = 1.$$

## Expected utility

The expected utility of a lottery $l = (p, 1-p)$ over two sums $x$ and $y$ is

$$Eu(p, 1-p) = p\,u(x) + (1-p)\,u(y).$$

# Application: finance

You speculate on the price of copper. You can buy

- a portfolio $x$ for 30K, which pays 80K if the copper price is at least \$2.53, and 0 otherwise;
- a portfolio $y$ for 30K, which pays 80K if the copper price is below \$2.47, and 0 otherwise;
- both portfolios, $x + y$.

**Net profits depending on the copper price**

| | price $\geq 2.53$ ($E_1$) | $2.53 >$ price $\geq 2.47$ ($E_2$) | $2.47 >$ price ($E_3$) |
|---|---|---|---|
| $x$ | 50K | $-30$K | $-30$K |
| $y$ | $-30$K | $-30$K | 50K |
| 0 (“neither”) | 0 | 0 | 0 |
| $x + y$ (“both”) | 20K | $-60$**K** | 20K |

# Other examples

**Elections**

You wonder whether to emigrate to (or invest in) a foreign country, whose immigration policy depends on the next election.

- With probability $p$, an anti-immigration candidate wins: you lose everything you spent on settling in, $-5000\,€$.
- Otherwise, a pro-immigration candidate wins: you get settlement aid and your income rises by $10\,000\,€$.

Your expected utility is $\quad p\,u(-5000) + (1-p)\,u(10\,000)$.

- **Horse racing:** your turn!
- **Medical treatment:** you are a doctor and must decide whether to prescribe a given drug.
- ...

# Representation by expected utility

As in the theory of choice without uncertainty, von Neumann and Morgenstern start from a primitive **preference** $\succsim$, here over **lotteries**. For example, with $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, if

$$l = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right) \prec (0, 1, 0) = l'$$

then the agent prefers (chooses) winning 2 € for sure over a $\frac{1}{2}$ chance of winning 1 € or 3 €.

**Question**

Which conditions on the agent's preference ensure that it is represented by an **expected utility**? That is, when is there a $u$ such that, for all $l = (p_1, \ldots, p_n)$ and $l' = (q_1, \ldots, q_n)$,

$$l \succsim l' \iff p_1 u(x_1) + \cdots + p_n u(x_n) \geq q_1 u(x_1) + \cdots + q_n u(x_n) \ ?$$

# The independence axiom

**Notation.** For lotteries $l = (p_1, \ldots, p_n)$, $l' = (q_1, \ldots, q_n)$ and a number $\alpha \in (0, 1]$, the **mixture** is

$$\alpha l + (1-\alpha) l' = \big(\alpha p_1 + (1-\alpha) q_1,\ \ldots,\ \alpha p_n + (1-\alpha) q_n\big).$$

**The axioms of von Neumann and Morgenstern**

- **Complete order:** the relation $\succsim$ is complete and transitive.
- **Independence:** if the agent prefers lottery $l_1$ to lottery $l_2$, then for every lottery $l_3$ and every number $\alpha$ between 0 and 1, she must prefer $\alpha l_1 + (1-\alpha) l_3$ to $\alpha l_2 + (1-\alpha) l_3$.
- **Continuity:** (roughly) if $l \succ l' \succ l''$, then I can build a lottery $\hat{l}$ between $l$ and $l''$, of the form $\hat{l} = p\, l + (1-p)\, l''$, such that the agent is indifferent between $\hat{l}$ and $l'$, i.e. $\hat{l} \sim l'$.

# The intuition behind independence

I offer you two lotteries:

- $l_1$: with probability $\frac{1}{2}$ you play a scratch card, otherwise you play roulette;
- $l_2$: with probability $\frac{1}{2}$ you play the national lottery, otherwise you play roulette.

**Independence says**

If you prefer the scratch card to the national lottery, then you must prefer $l_1$ to $l_2$: the common part (roulette) should not matter.

**A problem with the independence axiom**

It presupposes a **cardinal** utility.

# Risk aversion

**You choose between**

- 50 € for sure;
- 100 € with one chance in two, otherwise 0 €.

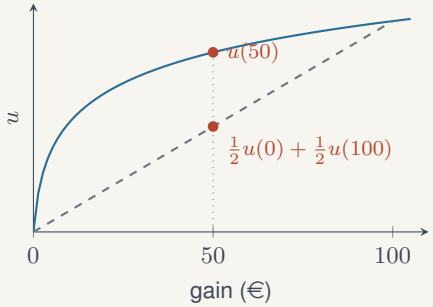
Which do you choose?

# Risk aversion

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Which do you choose?




To account for this, we can say that the utility $u$ **grows more and more slowly**:

$$Eu(p, 1-p) = p\,u(x) + (1-p)\,u(y).$$

Indeed, if $u(x) = \ln(x+1)$, then

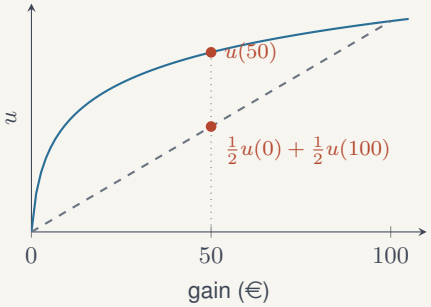
$$\ln(50+1) \approx 3.93 \;>\; \tfrac{1}{2}\ln(100+1) \approx 2.31.$$

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**Problem**

This amounts to saying that diminishing marginal utility and risk aversion are **the same thing**.

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# Savage: subjective probability (and utility)

Savage: from choices alone, recover both a subjective probability and a utility.

# Savage and subjective probabilities

**So far, two approaches**

1. Take **utility** as given and find the probabilities the agent uses (Ramsey and de Finetti).
2. Take **probabilities** as given and find the agent's utility (von Neumann and Morgenstern).

**Savage** (1954) shows that we can assume that *neither* ingredient is given and still obtain an expected utility with **subjective** probability *and* utility.

## Savage's framework

Simplifying somewhat, Savage starts from

- a set of **states of the world** $S = \{s_1, \ldots, s_n\}$;
- a set of **consequences** $C = \{c_1, \ldots, c_m\}$;
- a set of **acts** $F = \{f_1, \ldots, f_k\}$, where each act assigns a consequence to each state of the world: $f_2(s_1)$ is the consequence of act $f_2$ in state $s_1$.

**Example:** $S = \{s_1, s_2\}$, $C = \{c_1, c_2, c_3, c_4\}$

- $s_1 =$ "it rains"; $s_2 =$ "it is sunny";
- $c_1 =$ "wet, hands free"; $c_2 =$ "dry, but carrying an umbrella"; $c_3 =$ "dry, hands free"; $c_4 =$ "wet and carrying an umbrella";
- $f_1 =$ "I take the umbrella"; $f_2 =$ "I do not take the umbrella".

Then $f_1(s_1) = ?$, $f_1(s_2) = ?$, $f_2(s_1) = ?$, and $f_2(s_2) = ?$.

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Then $f_1(s_1) = c_2$, $f_1(s_2) = c_2$, $f_2(s_1) = c_1$, and $f_2(s_2) = c_3$.

## Savage's representation

Savage takes preferences over **acts**: for $f, g \in F$, $f \succsim g$ means that $f$ is preferred to (chosen over) $g$.

**Question**

When are there a utility $u$ and a probability $p$ such that

$$f \succsim g \iff \sum_{s \in S} u\big(f(s)\big)\, p(s) \geq \sum_{s \in S} u\big(g(s)\big)\, p(s) \ ?$$

**Remarks**

- The sum is not quite right: Savage's framework requires infinitely many states of the world (and hence of acts). But the idea is the same when everything is infinite.
- Savage uses seven axioms (the first: preferences are complete and transitive). The most important is the **sure-thing principle**, close to the independence axiom.

# Main axiom: the sure-thing principle

**Sure-thing principle**

If two acts have the same consequence in some state of the world (the "sure thing"), then this state is **irrelevant** for the agent's comparison of the two acts.
With two states $s_1, s_2$:

$$\left.\begin{array}{l} f(s_1) = g(s_1) \text{ and } f'(s_1) = g'(s_1) \\ f(s_2) = f'(s_2) \text{ and } g(s_2) = g'(s_2) \end{array}\right\} \Longrightarrow (f \succsim g \iff f' \succsim g')$$

**Interpretation ($s_1$ = economic growth, $s_2$ = no growth)**

- Not studying, $g$, earns me $g(s_1) = 100$ if there is growth and $g(s_2) = 50$ if not.
- Studying, $f$, also earns me $f(s_1) = 100$ if there is growth.
- The axiom says that my preference must depend **only** on the state with **no growth**: if there is growth, the consequence is the same either way.

## Bayesian learning

The subjective interpretation of probability lets us think about how **beliefs change** in the light of new information.

**Example**

Population: 50% women, 50% men. 80% of women voted Macron (vs. Le Pen), against 50% of men.

- Pick someone at random: what is the probability that she is a woman?

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- And if I tell you that this person voted Macron? Then

$$P(W \mid M) = \frac{P(M \mid W)\,P(W)}{P(M \mid W)\,P(W) + P(M \mid Men)\,P(Men)} = \frac{0.8 \times 0.5}{0.8 \times 0.5 + 0.5 \times 0.5} \approx \mathbf{62\%}$$

The prior belief (50%) is updated into a posterior belief (62%) by Bayes' rule.

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# Experimental challenges to expected utility

In the laboratory, people violate the axioms: Allais, framing, the conjunction fallacy, Ellsberg.

# The Allais paradox

**Choice 1**

$A$: 10 000 € with 100%
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$A \succ B$ but $D \succ C$: this **violates the independence axiom**.

# The framing effect (Tversky and Kahneman, 1981)

**Problem 1 [$N = 152$]**

Imagine that the United States is preparing for the outbreak of an epidemic from Asia, which is expected to kill 600 people. Two programs to fight the disease have been proposed:

- If program $A$ is adopted, **200 people will be saved**. (72% choose it)
- If program $B$ is adopted, 600 people will be saved with probability 1/3, and nobody will be saved with probability 2/3. (28%)

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**Problem 2 [$N = 155$]: same story**

- If program $C$ is adopted, **400 people will die**. (22% choose it)
- If program $D$ is adopted, nobody dies with probability 1/3, and all 600 die with probability 2/3. (78%)

# The framing effect: Kahneman and Tversky

**Two big ideas**

- The way a problem is **presented** affects the decision.
- People are **risk averse** when it comes to potential **gains**, and **risk seeking** when it comes to **losses**.

$A$ and $C$ describe the same outcome (200 saved $=$ 400 die), and so do $B$ and $D$. Yet the majority choice flips from the sure option ($A$) to the gamble ($D$).

# The conjunction fallacy (Tversky and Kahneman, 1983)

**Linda**

Linda is 31 years old. She is outspoken and very bright; she majored in philosophy. She is regularly involved in activism: for instance, she took part in anti-nuclear demonstrations.
Which is more probable?

1. Linda is a bank teller.
2. Linda is a bank teller and is active in the feminist movement.

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1. Linda is a bank teller.
2. Linda is a bank teller and is active in the feminist movement.

Most subjects answer 2, but the logical answer is **1**: a conjunction (“bank teller *and* feminist”) can never be more probable than one of its parts (“bank teller”).

# The Ellsberg paradox

An urn contains **90 balls**: **30 red**, and **60 green or blue** in unknown proportions. You choose between bets:

| | Red | Green | Blue |
|---|---|---|---|
| Bet $A$ | 100 | 0 | 0 |
| Bet $B$ | 0 | 100 | 0 |
| Bet $C$ | 100 | 0 | 100 |
| Bet $D$ | 0 | 100 | 100 |

**Typical lab observations**

- $A \succ B \Rightarrow P(R) > P(G)$
- $D \succ C \Rightarrow P(R) + P(B) < P(G) + P(B)$
  $\Rightarrow P(R) < P(G)$

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No probability can explain both choices. This **violates the sure-thing principle**, with **blue** playing the role of the sure thing.

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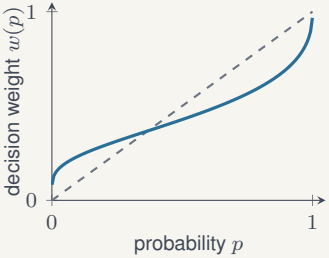
# Theories to account for the anomalies

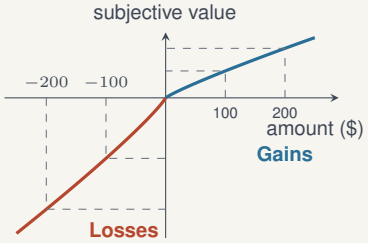
Prospect theory, regret, ambiguity, unawareness, inattention: economists rebuild the theory to fit the anomalies.

# Prospect theory: Kahneman and Tversky

## Two main ideas

- People have a **subjective sensitivity to probabilities** that leads them into errors: they **overweight small** probabilities and **underweight large** ones. So they maximize $w(p)\,u(R_1) + w(1-p)\,u(R_2)$.
- People have a **reference point** that separates what they see as a gain from what they see as a loss (this point can be shifted).



# Regret aversion: Loomes and Sugden (1982), Bell (1982)

The idea: people can **anticipate their regret**. Assuming $R_1 > R_2$, the utility they propose is

$$p\,u(R_1) + (1-p)\,u(R_2) + \mathcal{R}\big(u(R_1) - u(R_2)\big),$$

where $\mathcal{R}$ captures anticipated regret.

**This helps to**

- explain **intransitive** choices;
- explain why, in the lab, subjects tend to **overbid** in auctions;
- account for some financial behaviors.

**Empirical facts**

This bias is all the stronger when the stakes of the decision are high (it comes from people thinking **too much**, not too little).

# Ambiguity theory

The Ellsberg paradox showed that, under uncertainty, it is hard to claim that people reason with *one* probability.

To solve this paradox, some researchers propose that people

- have **several probabilities** in mind, a set $\Delta$;
- and choose the probability $p$ that gives the **worst scenario** (precautionary principle).

**MaxMin (Gilboa and Schmeidler, 1989)**

The agent evaluates each act with the probability that makes it look worst:

$$f \succsim g \iff \min_{p \in \Delta} \sum_{s \in S} u\big(f(s)\big)\, p(s) \;\geq\; \min_{p \in \Delta} \sum_{s \in S} u\big(g(s)\big)\, p(s).$$

# Unawareness theory

On closer inspection (closer than we can manage here), Savage's model assumes that people can **imagine every possible event** and assign it a probability.

- People may not know what will happen, but then they must **know that they do not know**. This is called **negative introspection**.

**Unawareness**

The theory of unawareness aims to describe situations in which people **do not know that they do not know** something.

(Mathematically, it quickly gets very hard.)

# Rational inattention

The idea: to make a good decision, we sometimes need **information**, but getting information can be **costly**.

- It may require studying, paying people to collect it, etc.

**Rational inattention**

A theory in which people **decide which information** they want to seek out.

# Some other theories related to beliefs

- **Narratives**: our beliefs are strongly influenced by the stories we tell ourselves or are told (see Shiller, 2019; Bénabou and Tirole, 2020).
- **Belief in a just world**: an idea of Bénabou and Tirole (2006), taken from the psychologist **Lerner**. People are more or less biased; some overestimate the role of *merit* in explaining a person's social trajectory, which has important political and economic consequences.
- **Cultural transmission**: models inspired by anthropology, in which parents raise their children so as to pass on their cultural traits (Bisin and Verdier, 2001).
- **Psychological reactance**, or the **backfire effect**: trying to convince someone that they are wrong can push them to dig deeper into their mistaken belief.

# Take-aways

1. Probability was born from **games of chance** (Pascal, Fermat): with it came **expectation**, and with the St. Petersburg paradox, **expected utility**.
2. Three readings of probability: **frequencies**, **logical** degrees of belief (Keynes), **subjective** beliefs revealed by bets (Ramsey, de Finetti). Without frequencies, we face **radical uncertainty** (Knight).
3. **Von Neumann–Morgenstern**: complete order + independence + continuity $\iff$ expected utility. **Savage**: subjective probability *and* utility, with the sure-thing principle.
4. Experiments challenge the theory: **Allais** (independence), **framing**, the **conjunction fallacy**, **Ellsberg** (sure-thing principle).
5. New theories respond: **prospect theory**, regret, ambiguity, unawareness, rational inattention.